College Algebra

Scottsdale Community College

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College Algebra
An Investigation of Functions

Custom Print for Scottsdale Community College

1st Edition

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Melonie Rasmussen

These books are also available to read free online at
http://www.opentextbookstore.com/precalc/ and http://www.stitz-zeager.com/Precalculus/Stitz_Zeager_Open_Source_Precalculus.html
If you want a printed copy, buying from the lulu.com is cheaper than printing yourself.

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About the Authors

David Lippman received his master's degree in mathematics from Western Washington University and has been teaching at Pierce College since Fall 2000.

Melonie Rasmussen also received her master's degree in mathematics from Western Washington University and has been teaching at Pierce College since Fall 2002. Prior to this Melonie taught for the Puyallup School district for 6 years after receiving her teaching credentials from Pacific Lutheran University.

We have both been long time advocates of open learning, open materials, and basically any idea that will reduce the cost of education for students. It started by supporting the college's calculator rental program, and running a book loan scholarship program. Eventually the frustration with the escalating costs of commercial text books and the online homework systems that charged for access led them to take action.

First, David developed IMathAS, open source online math homework software that runs WAMAP.org and MyOpenMath.com. Through this platform, we became integral parts of a vibrant sharing and learning community of teachers from around Washington State that support and contribute to WAMAP. Our pioneering efforts, supported by dozens of other dedicated faculty and financial support from the Transition Math Project, have led to a system used by thousands of students every quarter, saving hundreds of thousands of dollars over comparable commercial offerings.

David continued further and wrote his first open textbook, *Math in Society*, a math for liberal arts majors book, after being frustrated by students having to pay $100+ for a textbook for a terminal course. Together, frustrated by both cost and the style of commercial texts, we began writing *PreCalculus: An Investigation of Functions* in 2010.

Acknowledgements

We would like to thank the following for their generous support and feedback.

- The community of WAMAP users and developers for creating a majority of the homework content used in our online homework sets.

- Pierce College students in our Fall 2010 - Summer 2011 Math 141 and Math 142 classes for helping correct typos, identifying videos related to the homework, and being our willing test subjects.

- The Open Course Library Project for providing the support needed to produce a full course package for these courses.

- Tophe Anderson, Chris Willett, and Vauhn Foster-Grahler for reviewing the course and giving feedback and suggestions.

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- Kevin Dimond for his work on indexing the book and creating PowerPoint slides.

Preface

Over the years, when reviewing books we found that many had been mainstreamed by the publishers in an effort to appeal to everyone, leaving them with very little character. There were only a handful of books that had the conceptual and application driven focus we liked, and most of those were lacking in other aspects we cared about, like providing students sufficient examples and practice of basic skills. The largest frustration, however, was the never ending escalation of cost and being forced into new editions every three years. We began researching open textbooks, however the ability for those books to be adapted, remixed, or printed were often limited by the types of licenses, or didn't approach the material the way we wanted.

This book is available online for free, in both Word and PDF format. You are free to change the wording, add materials and sections or take them away. We welcome feedback, comments and suggestions for future development at (insert an email address here). Additionally, if you add a section, chapter or problems, we would love to hear from you and possibly add your materials so everyone can benefit.

In writing this book, our focus was on the story of functions. We begin with function notation, a basic toolkit of functions, and the basic operation with functions: composition and transformation. Building from these basic functions, as each new family of functions is introduced we explore the important features of the function: its graph, domain and range, intercepts, and asymptotes. The exploration then moves to evaluating and solving equations involving the function, finding inverses, and culminates with modeling using the function.

The "rule of four" is integrated throughout - looking at the functions verbally, graphically, numerically, as well as algebraically. We feel that using the "rule of four" gives students the tools they need to approach new problems from various angles. Often the "story problems of life" do not always come packaged in a neat equation. Being able to think critically, see the parts and build a table or graph a trend, helps us change the words into meaningful and measurable functions that model the world around us.

There is nothing we hate more than a chapter on exponential equations that begins "Exponential functions are functions that have the form $f(x)=a^x$." As each family of functions is introduced, we motivate the topic by looking at how the function arises from life scenarios or from modeling. Also, we feel it is important that precalculus be the bridge in level of thinking between algebra and calculus. In algebra, it is common to see numerous examples with very similar homework exercises, encouraging the student to mimic the examples. Precalculus provides a link that takes students from the basic plug & chug of formulaic calculations towards building an understanding that equations and formulas have deeper meaning and purpose. While you will find examples and similar exercises for the basic skills in this book, you will also find examples of multistep problem solving along with exercises in multistep problem solving. Often times these exercises will not exactly mimic the exercises, forcing the students to employ their critical thinking skills and apply the skills they've learned to new situations. By developing students' critical thinking and problem solving skills this course prepares students for the rigors of Calculus.

While we followed a fairly standard ordering of material in the first half of the book, we took some liberties in the trig portion of the book. It is our opinion that there is no need to separate unit circle trig from triangle trig, and instead integrated them in the first chapter. Identities are introduced in the first chapter, and revisited throughout. Likewise, solving is introduced in the second chapter and revisited more extensively in the third chapter. As with the first part of the book, an emphasis is placed on motivating the concepts and on modeling and interpretation.

How To Be Successful In This Course

This is not a high school math course, although for some of you the content may seem familiar. There are key differences to what you will learn here, how quickly you will be required to learn it and how much work will be required of you.

You will no longer be shown a technique and be asked to mimic it repetitively as the only way to prove learning. Not only will you be required to master the technique, but you will also be required to extend that knowledge to new situations and build bridges between the material at hand and the next topic, making the course highly cumulative.

As a rule of thumb, for each hour you spend in class, you should expect this course will require an average of 2 hours of out of class focused study. This means that some of you with a stronger background in mathematics may take less, but if you have a weaker background or any math anxiety it will take you more.

Notice how this is the equivalent of having a part time job, and if you are taking a fulltime load of courses as many college students do, this equates to more than a full time job. If you must work, raise a family and take a full load of courses all at the same time, we recommend that you get a head start & get organized as soon as possible. We also recommend that you spread out your learning into daily chunks and avoid trying to cram or learn material quickly before an exam.

To be prepared, read through the material before it is covered in and note or highlight the material that is new or confusing. The instructor's lecture and activities should not be the first exposure to the material. As you read, test your understanding with the Try it Now problems in the book. If you can't figure one out, try again after class, and ask for help if you still can't get it.

As soon as possible after the class session recap the days lecture or activities into a meaningful format to provide a third exposure to the material. You could summarize your notes into a list of key points, or reread your notes and try to work examples done in class without referring back to your notes. Next, begin any assigned homework. The next day, if the instructor provides the opportunity to clarify topics or ask questions, do not be afraid to ask. If you are afraid to ask, then you are not getting your money's worth! If the instructor does not provide this opportunity, be prepared to go to a tutoring center or build a peer study group. Put in quality effort and time and you can get quality results.

Lastly, if you feel like you do not understand a topic. Don't wait, ASK FOR HELP!

ASK: **A**sk a teacher or tutor, **S**earch for ancillaries, **K**eep a detailed list of questions
FOR: **F**ind additional resources, **O**rganize the material, **R**esearch other learning options
HELP: **H**ave a support network, **E**xamine your weaknesses, **L**ist specific examples & **P**ractice

Best of luck learning! We hope you like the course & love the price.
David & Melonie

Table of Contents

This table of contents is specifically for the custom print being used by Scottsdale Community College. The page numbers refer to the large page numbers in the upper right hand corner which begin on page 11.

Welcome to the college algebra text for Scottsdale Community College! We are combining two open texts to meet our course needs. We have the Precalculus text by David Lippman and Melonie Rasumussen which will comprise most of the course. Then there is a supplementary section beginning on page 312 from the College Algebra Book by Carl Stitz and Jeff Zeager. There is a table of contents for each section as well as homework problems, solutions and indexes.

This custom textbook combines material from two open source books.
Precalculus: An Investigation of Functions by David Lippman and Melonie Rasmussen
http://www.opentextbookstore.com/precalc/
And
College Algebra by Stitz and Zeager
http://www.stitz-zeager.com/Precalculus/Stitz_Zeager_Open_Source_Precalculus.html
Both of these texts are free to use and you may visit their websites for various supplements.

Chapter 1: Functions

Section 1.1 Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "if I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship: in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine height from a given age, that would be problematic, since most people maintain the same height for many years.

Function

Function: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

One-to-One Function

Sometimes in a relationship each input corresponds to exactly one output, and every output corresponds to exactly one input. We call this kind of relationship a **one-to-one function**.

From Example 3, *if* each unique percentage corresponds to one unique decimal grade point and each unique decimal grade point corresponds to one unique percentage then it is a one-to-one function.

Try it Now

Let's consider bank account information.

1. Is your balance a function of your bank account number?
(if you input a bank account number does it make sense that the output is your balance?)

2. Is your bank account number a function of your balance?
(if you input a balance does it make sense that the output is your bank account number?)

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable *h* to represent height and we could use the descriptive variable *a* to represent age.

"height is a function of age"	if we name the function *f* we write
"*h* is *f* of *a*"	or more simply
$h = f(a)$	we could instead name the function *h* and write
$h(a)$	which is read "*h* of *a*"

Remember we can use any variable to name the function; the notation $h(a)$ shows us that *h* depends on *a*. The value "*a*" must be put into the function "*h*" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Function Notation

The notation output = *f*(input) defines a function named *f*. This would be read "output is *f* of input"

Example 4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function *f*, we could write "days = *f*(month)" or $d = f(m)$. If we simply name the function *d*, we could write $d(m)$

For example, *d*(March) = 31, since March has 31 days. The notation $d(m)$ reminds us that the number of days, *d* (the output) is dependent on the name of the month, *m* (the input)

Example 5

A function $N = f(y)$ gives the number of police officers, *N*, in a town in year *y*. What does $f(2005) = 300$ tell us?

When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (*y*). The output value is 300, the number of police officers (*N*), a value for the output quantity. Remember $N=f(y)$. So this tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words, as we did in the last few examples, tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, *m*	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, *D*	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function $Q = g(n)$. Remember this notation tells us g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) *a*, age in years	5	5	6	7	8	9	10
(output) *h*, height inches	40	42	44	47	50	52	54

Example 6

Which of these tables define a function (if any), are any of them one-to-one?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

Only the first table is one-to-one; it is both a function, and each output corresponds to exactly one input. Although table 2 is a function, because each input corresponds to exactly one output, each output does not correspond to exactly one input so this function is not one-to-one. Table 3 is not even a function and so we don't even need to consider if it is a one-to-one function.

Try it Now

3. If each percentage earned translated to one grade point average would this be a function?

Solving & Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving a function is what we do when we know an output, and use the function to determine the inputs that would produce those outputs. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 7

Using the table shown, where $Q=g(n)$

a) Evaluate $g(3)$

n	1	2	3	4	5
Q	8	6	7	6	8

Evaluating $g(3)$ (read: "*g* of 3")
means that we need to determine the output value, Q, of the function g given the input value of n=3. Looking at the table, we see the output corresponding to n=3 is Q=7, allowing us to conclude $g(3) = 7$.

b) Solve $g(n) = 6$

Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

When we input 2 into the function g, our output is $Q = 6$

When we input 4 into the function g, our output is also $Q = 6$

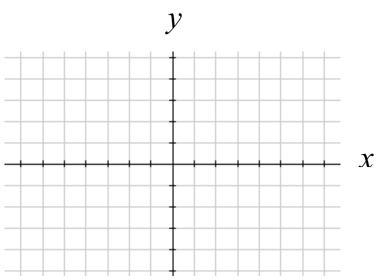
Try it Now

4. Using the function in Example 7, evaluate $g(4)$

Graphs as Functions

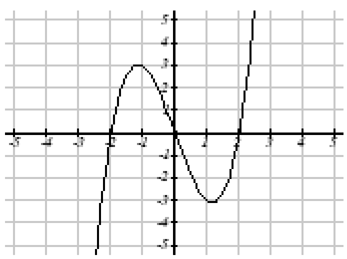
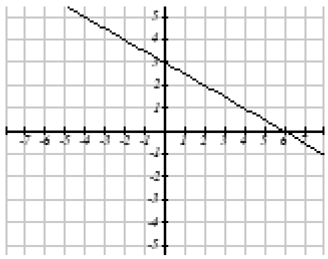
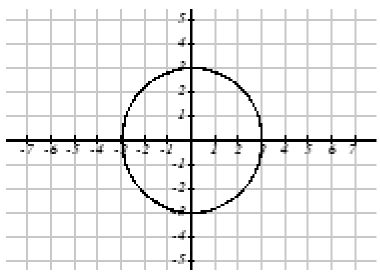
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has *y* on the vertical axis and *x* on the horizontal axis, and we say *y* is a function of *x*, or $y = f(x)$ when the function is named *f*.



Example 8

Which of these graphs defines a function $y=f(x)$? Which of these graphs defines a one-to-one function?

Looking at the three graphs above, the first two define a function $y=f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the *y*-value of the graph. The 3rd graph on does not define a function $y=f(x)$ since some input values, such as $x=2$, correspond with more than one output value.

Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -2 and 2.3

Graph 2 is a one-to-one function, each input corresponds to exactly one output, and every output corresponds to exactly one input.

Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define the vertical output as a function of the horizontal input.

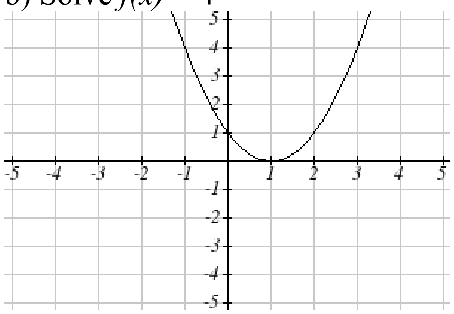
Horizontal Line Test

Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

Evaluating a function using a graph requires taking the given input, and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output, and looking on the graph to determine the corresponding input.

Example 9

Given the graph below,
a) Evaluate $f(2)$
b) Solve $f(x) = 4$



a) To evaluate $f(2)$, we find the input of $x=2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of $y=1$. So $f(2) = 1$

b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$
This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4, shows us that this function is not one-to-one.

Try it Now

5. Using the graph from example 9, solve $f(x)=1$

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as p = [something involving n].

$2n + 6p = 12$ subtract $2n$ from both sides
$6p = 12 - 2n$ divide both sides by 6 and simplify

$$p = \frac{12 - 2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as p=, we can now express p as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 11

Express the relationship $x^2 + y^2 = 1$ as a function $y = f(x)$ if possible.

If we try to solve for y in this equation:

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input that would produce that output.

Example 12

Given the function $k(t) = t^3 + 2$

a) Evaluate $k(2)$
b) Solve $k(t) = 1$

a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable t, then simplify

$k(2) = 2^3 + 2$

$k(2) = 8 + 2$

So $k(2) = 10$

b) To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output

$k(t) = 1$	substitute the original formula $k(t) = t^3 + 2$
$t^3 + 2 = 1$	subtract 2 from each side
$t^3 = -1$	take the cube root of each side
$t = -1$	

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t) = 1$ is true when $t = -1$.

$k(-1) = (-1)^3 + 2$

$= -1 + 2$

$= 1$ which was the desired result.

Example 13

Given the function $h(p) = p^2 + 2p$

a) Evaluate $h(4)$
b) Solve $h(p) = 3$

To evaluate $h(4)$ we substitute the value 4 in for the input variable p in the given function.

a) $h(4) = (4)^2 + 2(4)$

$= 16 + 8$

$= 24$

b) $h(p) = 3$	Substitute the original function $h(p) = p^2 + 2p$
$p^2 + 2p = 3$	This is quadratic, so we can rearrange the equation to get it = 0
$p^2 + 2p - 3 = 0$	subtract 3 from each side
$p^2 + 2p - 3 = 0$	this is factorable, so we factor it

$(p+3)(p-1)=0$

By the zero factor theorem since $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0) and so we solve both equations for *p*, finding *p* = -3 from the first equation and p = 1 from the second equation.

This gives us the solution: *h(p)* = 3 when *p* = 1 or *p* = -3

We found two solutions in this case, which tells us this function is not one-to-one.

Try it Now

6. Given the function $g(m)=\sqrt{m-4}$

a. Evaluate *g*(5)

b. Solve *g(m)* = 2

Basic Toolkit Functions

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our "toolkit of functions" – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use *x* as the input variable and *f(x)* as the output variable.

Toolkit Functions

Linear

- Constant: $f(x)=c$, where *c* is a constant (number)
- Identity: $f(x)=x$

Absolute Value: $f(x)=|x|$

Power

- Quadratic: $f(x)=x^2$
- Cubic: $f(x)=x^3$
- Reciprocal: $f(x)=\dfrac{1}{x}$
- Reciprocal squared: $f(x)=\dfrac{1}{x^2}$
- Square root: $f(x)=\sqrt[2]{x}=\sqrt{x}$
- Cube root: $f(x)=\sqrt[3]{x}$

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

Not every important equation can be written where $y = f(x)$. An example of this is the equation of a circle. Recall the distance formula for the distance between two points:

$$dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

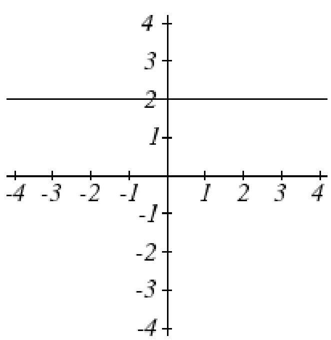
A circle with radius r with center at (h, k) can be described as all points (x, y) a distance of r from the center, so using the distance formula, $r = \sqrt{(x-h)^2 + (y-k)^2}$, giving

Equation of a circle

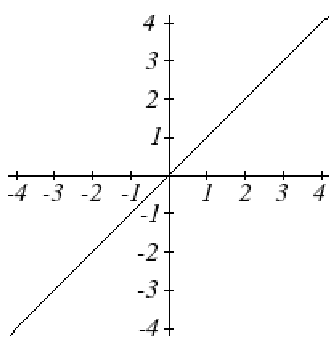
A circle with radius r with center (h, k) has equation $r^2 = (x-h)^2 + (y-k)^2$

Graphs of the Toolkit Functions

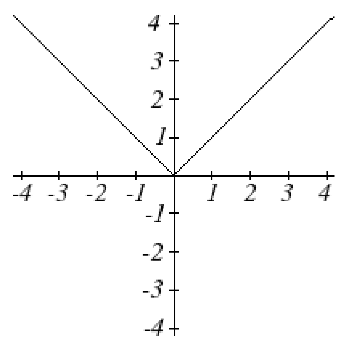
Constant Function: $f(x) = 2$



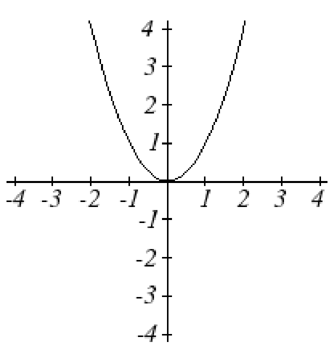
Identity: $f(x) = x$



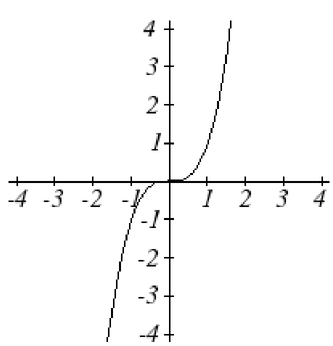
Absolute Value: $f(x) = |x|$



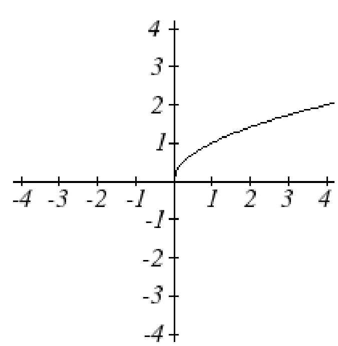
Quadratic: $f(x) = x^2$



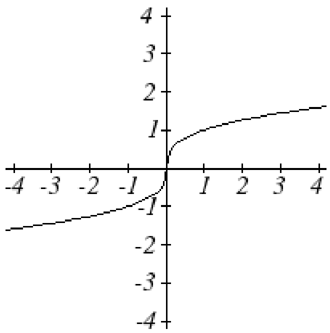
Cubic: $f(x) = x^3$



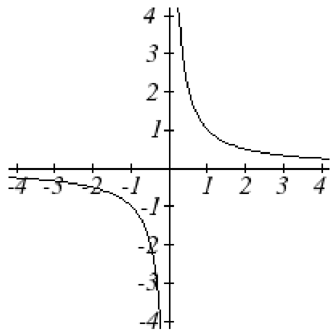
Square root: $f(x) = \sqrt{x}$



Cube root: $f(x)=\sqrt[3]{x}$

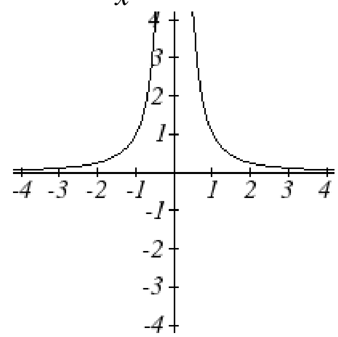


Reciprocal: $f(x)=\dfrac{1}{x}$



Reciprocal squared:

$f(x)=\dfrac{1}{x^2}$



Important Topics of this Section

Definition of a function
Input (independent variable)
Output (dependent variable)
Definition of a one-to-one function
Function notation
Descriptive variables
Functions in words, tables, graphs & formulas
Vertical line test
Horizontal line test
Evaluating a function at a specific input value
Solving a function given a specific output value
Toolkit Functions

Try it Now Answers

1. Yes
2. No
3. Yes
4. $Q=g(4)=6$
5. $x = 0$ or $x = 2$
6. a. $g(5)=1$ b. $m = 8$

Section 1.1 Exercises

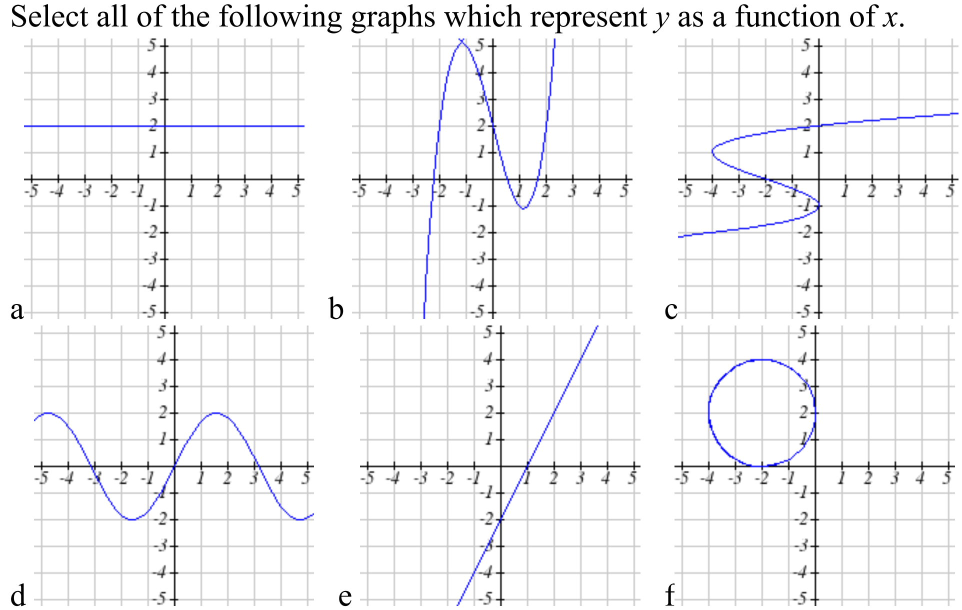
1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
    a. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100) = 1$

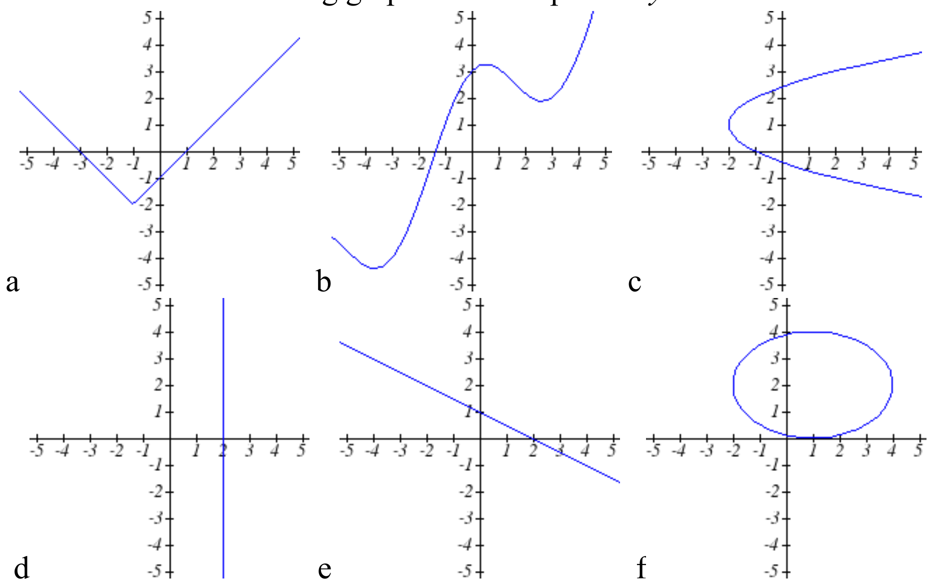
3. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$ b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$ b. $h(2) = 350$

5. Select all of the following graphs which represent y as a function of x.



6. Select all of the following graphs which represent y as a function of x.



7. Select all of the following tables which represent y as a function of x.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent y as a function of x.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent y as a function of x.

a.

x	**y**
0	-2
3	1
4	6
8	9
3	1

b.

x	**y**
-1	-4
2	3
5	4
8	7
12	11

c.

x	**y**
0	-5
3	1
3	4
9	8
16	13

d.

x	**y**
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent y as a function of x.

a.

x	**y**
-4	-2
3	2
6	4
9	7
12	16

b.

x	**y**
-5	-3
2	1
2	4
7	9
11	10

c.

x	**y**
-1	-3
1	2
5	4
9	8
1	2

d.

x	**y**
-1	-5
3	1
5	1
8	7
14	12

11. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	3	8	12
y	4	7	7

b.

x	3	8	12
y	4	7	13

c.

x	3	8	8
y	4	7	13

12. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	2	8	8
y	5	6	13

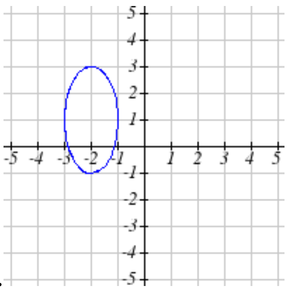
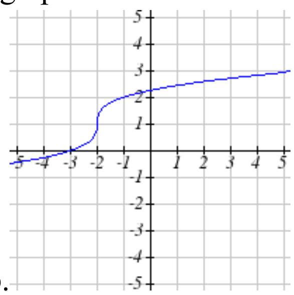
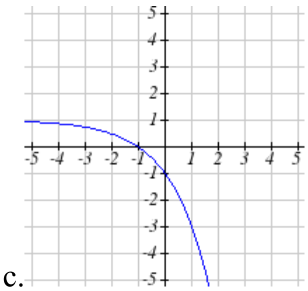
b.

x	2	8	14
y	5	6	6

c.

x	2	8	14
y	5	6	13

13. Select all of the following graphs which are **one-to-one functions**.


a.

b.

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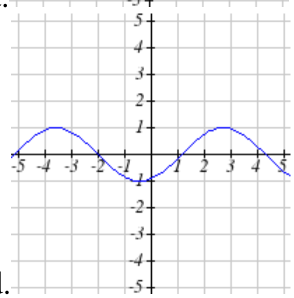
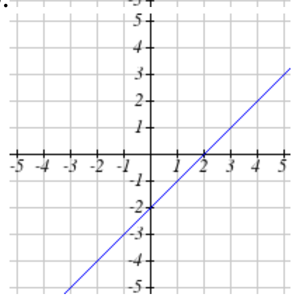
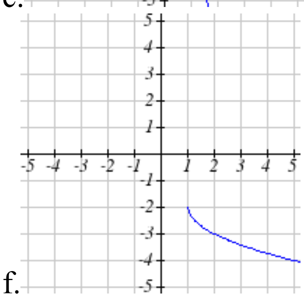
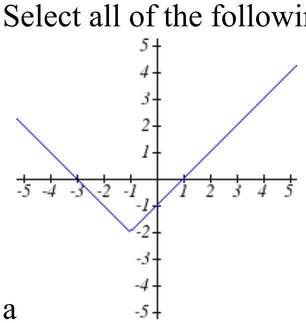
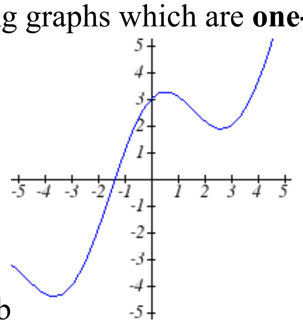
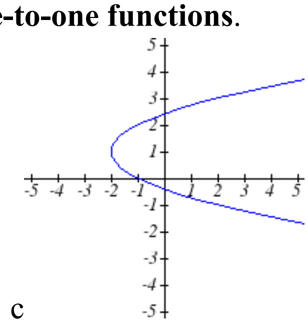
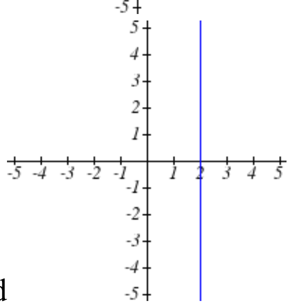
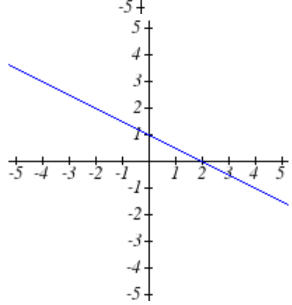
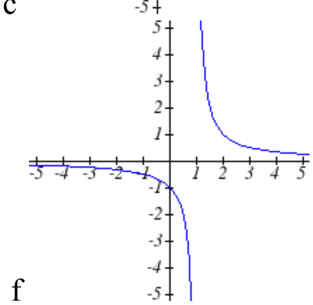
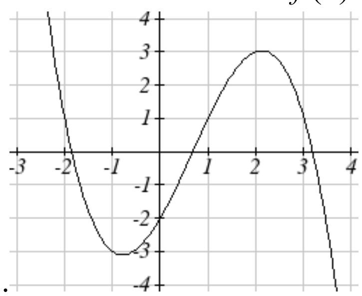
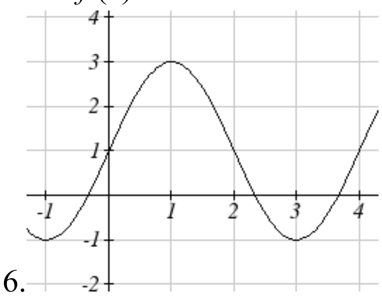

d.

e.

f.

14. Select all of the following graphs which are **one-to-one functions**.

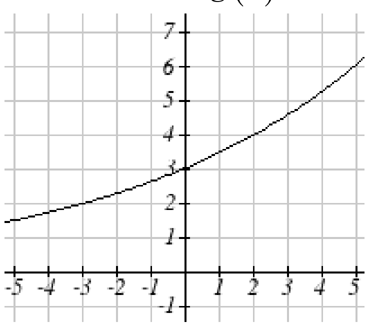

a

b

c

d

e

f

Given the each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

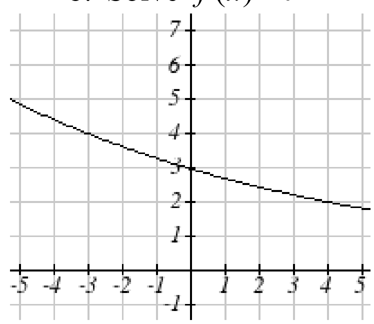
15. 

16. 

17. Given the function $g(x)$ graphed here,
    a. Evaluate $g(2)$
    b. Solve $g(x)=2$



18. Given the function $f(x)$ graphed here.
    a. Evaluate $f(4)$
    b. Solve $f(x)=4$



19. Based on the table below,
    a. Evaluate $f(3)$ b. Solve $f(x)=1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

20. Based on the table below,
    a. Evaluate $f(8)$ b. Solve $f(x)=7$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

For each of the following functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

21. $f(x)=4-2x$
22. $f(x)=8-3x$
23. $f(x)=8x^2-7x+3$
24. $f(x)=6x^2-7x+4$
25. $f(x)=-x^3+2x$
26. $f(x)=5x^4+x^2$
27. $f(x)=3+\sqrt{x+3}$
28. $f(x)=4-\sqrt[3]{x-2}$
29. $f(x)=(x-2)(x+3)$
30. $f(x)=(x+3)(x-1)^2$
31. $f(x)=\dfrac{x-3}{x+1}$
32. $f(x)=\dfrac{x-2}{x+2}$
33. $f(x)=2^x$
34. $f(x)=3^x$

35. Suppose $f(x)=x^2+8x-4$. Compute the following:

a. $f(-1)+f(1)$ b. $f(-1)-f(1)$

36. Suppose $f(x)=x^2+x+3$. Compute the following:

a. $f(-2)+f(4)$ b. $f(-2)-f(4)$

37. Let $f(t)=3t+5$

a. Evaluate $f(0)$ b. Solve $f(t)=0$

38. Let $g(p)=6-2p$

a. Evaluate $g(0)$ b. Solve $g(p)=0$

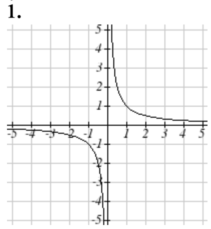
39. Match each function name with its equation.

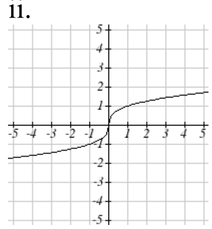
a. $y=x$
b. $y=x^3$
c. $y=\sqrt[3]{x}$
d. $y=\dfrac{1}{x}$
e. $y=x^2$
f. $y=\sqrt{x}$
g. $y=|x|$
h. $y=\dfrac{1}{x^2}$

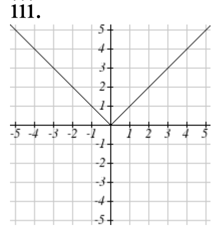
i. Cube root
ii. Reciprocal
iii. Linear
iv. Square Root
v. Absolute Value
vi. Quadratic
vii. Reciprocal Squared
viii. Cubic

40. Match each graph with its equation.

a. $y=x$
b. $y=x^3$
c. $y=\sqrt[3]{x}$
d. $y=\dfrac{1}{x}$
e. $y=x^2$
f. $y=\sqrt{x}$
g. $y=|x|$
h. $y=\dfrac{1}{x^2}$

i.


ii.


iii.


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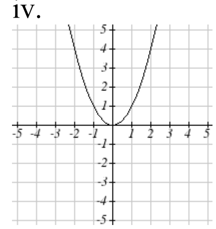
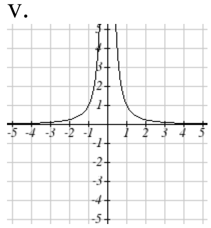
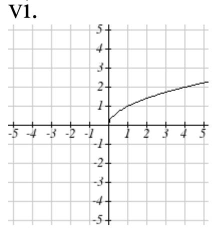
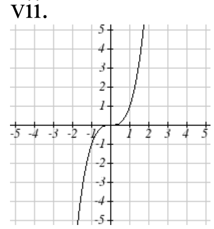


v.


vi.


vii.


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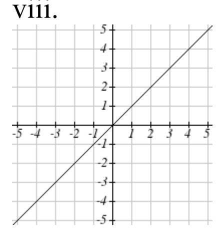


41. Match each table with its equation.

a. $y = x^2$
b. $y = x$
c. $y = \sqrt{x}$
d. $y = 1/x$
e. $y = |x|$
f. $y = x^3$

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

42. Match each equation with its table

a. Quadratic
b. Absolute Value
c. Square Root
d. Linear
e. Cubic
f. Reciprocal

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

43. Write the equation of the circle centered at $(3, -9)$ with radius 6.

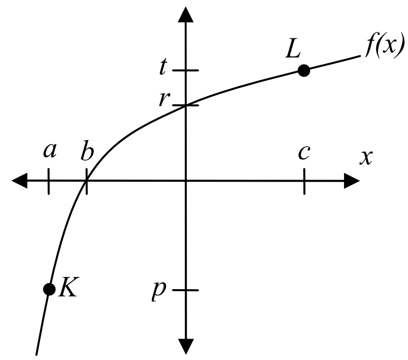
44. Write the equation of the circle centered at $(9, -8)$ with radius 11.

45. Sketch a reasonable graph for each of the following functions. [UW]

a. Height of a person depending on age.
b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
c. The amount of postage you must put on a first class letter, depending on the weight of the letter.

46. Sketch a reasonable graph for each of the following functions. [UW]
   a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
   b. You height above the water level in a swimming pool after you dive off the high board.
   c. The percentage of dates and names you'll remember for a history test, depending on the time you study

47. Using the graph shown,
   a. Evaluate $f(c)$
   b. Solve $f(x) = p$
   c. Suppose $f(b) = z$. Find $f(z)$
   d. What are the coordinates of points L and K?



48. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function s = d(t) which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]

   a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.

   b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

   c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of “distance from Padelford” vs. “time” for the both Angela and Dave.

f. Suppose you want to sketch the graph of a new function $s = g(t)$ that keeps track of Dave’s distance s from Gould Hall at time t. How would your graphs change in (a)-(e)?

Section 1.2 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between tree circumference and height.

Circumference, c	1.7	2.5	5.5	8.2	13.7
Height, h	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function

Domain and Range

Domain: The set of possible input values to a function
Range: The set of possible output values of a function

Example 1

Using the tree table above, determine a reasonable domain and range.

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is 163 feet[1]. With this information we would say a reasonable domain is $0 < c \le 163$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \le 379$ feet.

Example 2

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter[2], as shown in the table below. Determine the domain and range.

---

[1] http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010
[2] http://www.usps.com/prices/first-class-mail-prices.htm, retrieved July 19, 2010

Letters	
Weight not Over	Price
1 ounce	\$0.44
2 ounces	\$0.61
3 ounces	\$0.78
3.5 ounces	\$0.95

Suppose we notate Weight by w and Price by p, and set up a function named P, where Price, p is a function of Weight, w. $p = P(w)$.

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be $0 < w \leq 3.5$. Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is p = \$0.44, \$0.61, \$0.78, or \$0.95.

Try it Now

1. The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$; $0 < w \leq 3.5$ and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, p = \$0.44, \$0.61, \$0.78, or \$0.95. These numbers represent a set of specific values: {0.44, 0.61, 0.78, 0.95}

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable x that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x \mid 10 \le x < 30\}$. The curly brackets {} are read as "the set of", and the vertical bar | is read as "such that", so altogether we would read $\{x \mid 10 \le x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to *x* and *x* is less than 30."

When describing ranges in set-builder notation, we could similarly write something like $\{f(x) \mid 0 < f(x) < 100\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h \mid 0 < h \le 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t \mid t \text{ is a real number}\}$, or $\{t \mid t \in \mathrm{R}\}$, read as "the set of *t*-values such that *t* is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than", and square brackets are used for "less than or equal to". The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	**Set Builder Notation**	**Interval notation**
$5 < h \le 10$	$\{h \mid 5 < h \le 10\}$	(5, 10]
$5 \le h < 10$	$\{h \mid 5 \le h < 10\}$	[5, 10)
$5 < h < 10$	$\{h \mid 5 < h < 10\}$	(5, 10)
$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
$h \ge 10$	$\{h \mid h \ge 10\}$	$[10, \infty)$
all real numbers	$\{h \mid h \in \mathrm{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Example 3

Describe the intervals of values shown on the line graph below using set builder and interval notations.

To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5"

As an inequality it is $1 \le x \le 3$ or $x > 5$

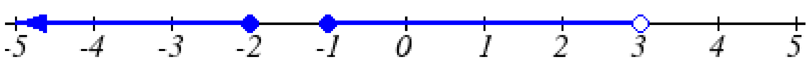
In set builder notation $\{x \mid 1 \le x \le 3 \text{ or } x > 5\}$

In interval notation, $[1,3] \cup (5,\infty)$

Remember when writing or reading interval notation:
Using a square bracket [means the start value is included in the set
Using a parenthesis (means the start value is not included in the set

Try it Now

2. Given the following interval write its meaning in words, set builder notation and interval notation.



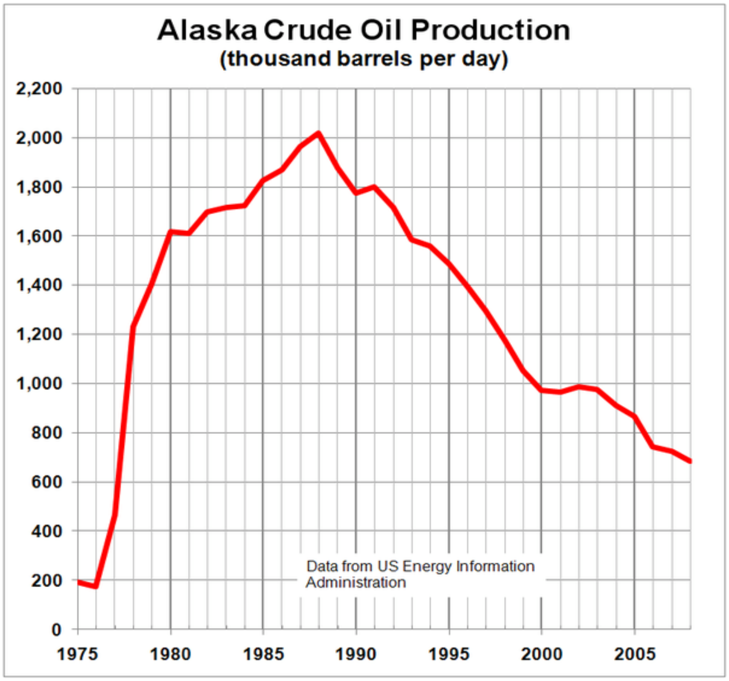
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 4

Determine the domain and range of the graph below.



In the graph above[3], the input quantity along the horizontal axis appears to be "year", which we could notate with the variable y. The output is "thousands of barrels of oil per day", which we might notate with the variable b, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \le y \le 2008$, and the range is approximately $180 \le b \le 2010$.

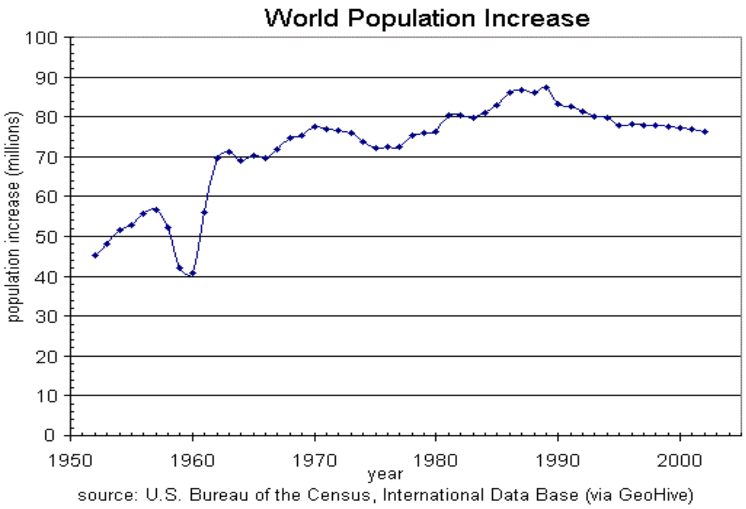
In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

Remember that as in the previous example, x and y are not always the input & output variables. Using descriptive variables is an important tool to remembering the context of the problem.

[3] http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG, CC-BY-SA, July 19, 2010

Try it Now

3. Given the graph below write the domain and range in interval notation



Domains and Ranges of the Toolkit functions

We will now return to our set of toolkit functions to note the domain and range of them. If you have completed the project assignment in Section 1.2 you can compare your reasonable input values and corresponding output values to the domain and range values listed below.

Constant Function, $f(x) = c$

The domain here is not restricted; x can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.
Domain: $(-\infty, \infty)$
Range: $[c]$
Since there is only one output value, we list it by itself in square brackets.

Identity Function, $f(x) = x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Quadratic Function, $f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Multiplying a negative or positive number by itself can only yield positive outputs

Cubic Function, $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Reciprocal, $f(x) = \dfrac{1}{x}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
We cannot divide by 0 so we must exclude 0 from the domain.

Reciprocal squared, $f(x) = \dfrac{1}{x^2}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
We cannot divide by 0 so we must exclude 0 from the domain.

Cube Root, $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Square Root, $f(x) = \sqrt[2]{x}$, commonly just written as, $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.

Absolute Value Function, $f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Since Absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

Piecewise Functions
In the tool kit functions we introduced the absolute value function $f(x) = |x|$.
With a domain of all real numbers and a range of values greater than or equal to 0, the absolute value has been defined as the magnitude or modulus of a number, a real number value regardless of sign, the size of the number, or the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value the output is unchanged
$f(x) = x \quad \text{if} \quad x \geq 0$

If we input a negative value the sign must change from negative to positive.
$f(x) = -x \quad \text{if} \quad x < 0$ since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x)=\begin{cases}\text{formula 1} & \text{if} \quad \text{domain to use function 1}\\ \text{formula 2} & \text{if} \quad \text{domain to use function 2}\\ \text{formula 3} & \text{if} \quad \text{domain to use function 3}\end{cases}$$

Example 5

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, *n*, to the cost, *C*.

To set up this function, two different formulas would be needed. $C = 5n$ would work for *n* values under 10, and $C = 50$ would work for values of *n* ten or greater. Notating this:

$$C(n)=\begin{cases}5n & if \quad 0<n<10\\ 50 & if \quad n\geq 10\end{cases}$$

Example 6

A cell phone company uses the function below to determine the cost, *C*, in dollars for *g* gigabytes of data transfer.

$$C(g)=\begin{cases}25 & if \quad 0<g<2\\ 25+10(g-2) & if \quad g\geq 2\end{cases}$$

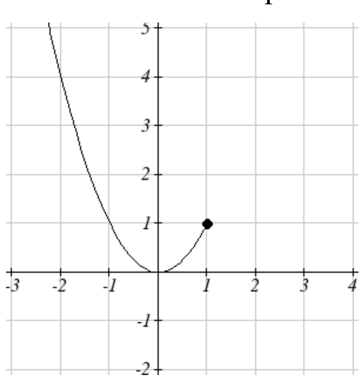
Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.
To find the cost of using 1.5 gigabytes of data, *C*(1.5), we first look to see which function's domain our input falls in. Since 1.5 is less than 2, we use the first function, giving *C*(1.5) = \$25.

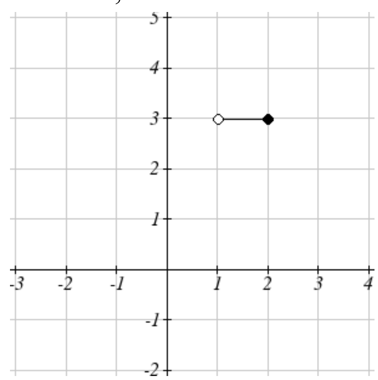
The find the cost of using 4 gigabytes of data, *C*(4), we see that our input of 4 is greater than 2, so we'll use the second function. *C*(4) = 25 + 10(4-2) = \$45.

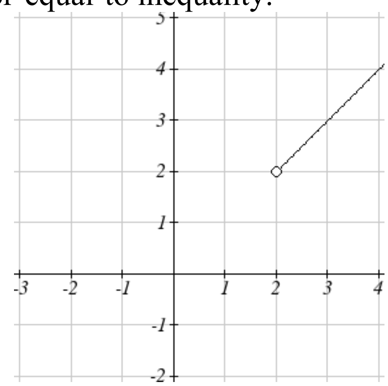
Example 7

Sketch a graph of the function $f(x)=\begin{cases} x^2 & if \quad x \le 1 \\ 3 & if \quad 1 < x \le 2 \\ x & if \quad x > 2 \end{cases}$

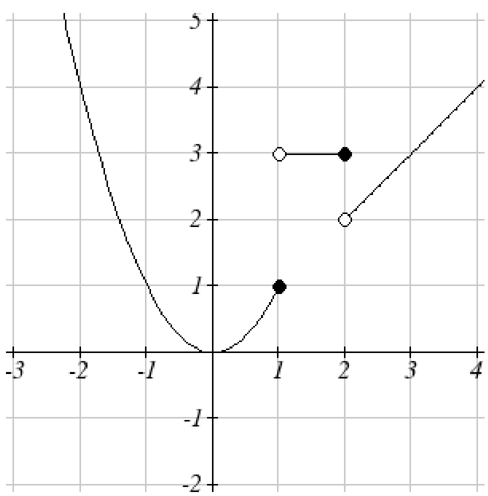
Since each of the component functions are from our library of Toolkit functions, we know their shapes. We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality.







Now that we have each piece individually, we combine them onto the same graph:



Try it Now

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit[4]. Write a piecewise defined function for the total tuition, *T*, at Pierce College 2009-2010 as a function of the number of credits taken, *c*. Be sure to consider reasonable domain and range.

Important Topics of this Section

Definition of domain
Definition of range
Inequalities
Interval notation
Set builder notation
Domain and Range from graphs
Domain and Range of toolkit functions
Piecewise defined functions

Try it Now Answers

1. Domain; y = years $[1960,2010]$; Range, p = population, $[100,1400]$

2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3
   b. $\{x \mid x \le -2 \text{ or } -1 \le x < 3\}$
   c. $(-\infty,-2] \cup [-1,3)$

3. Domain; y=years $[1952,2002]$; Range, p=population in millions, $[40,88]$

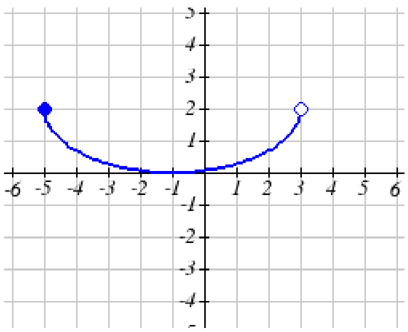
4. $T(c) = \begin{cases} 89.5c & \text{if} \quad c \le 10 \\ 895 + 33(c-10) & \text{if} \quad 10 < c \le 18 \\ 1159 + 73(c-18) & \text{if} \quad c > 18 \end{cases}$ Tuition, *T*, as a function of credits, *c*.

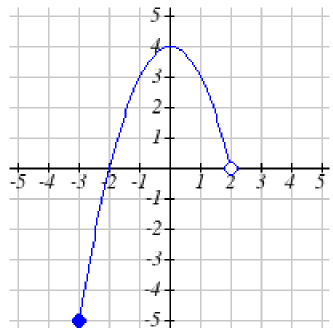
Reasonable domain should be whole numbers 0 to (answers may vary)
Reasonable range should be \$0 – (answers may vary)

[4] https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf, retrieved August 6, 2010

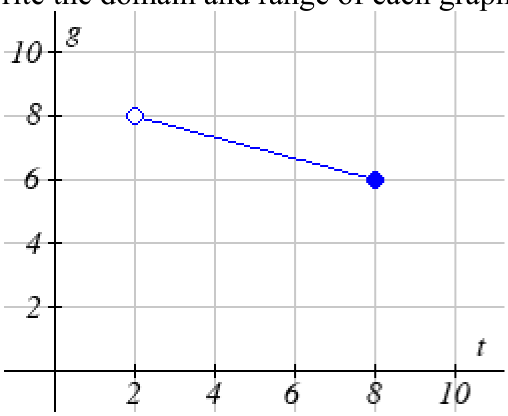
Section 1.2 Exercises

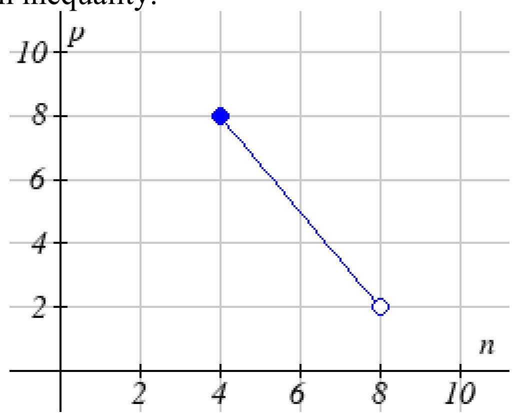
Write the domain and range of the function using interval notation.

1. 

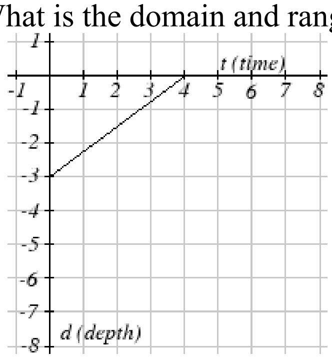
2. 

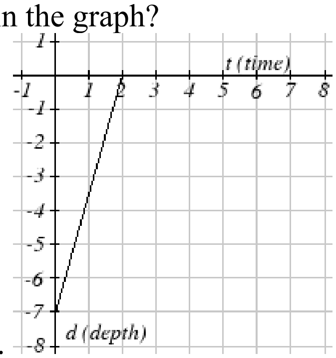
Write the domain and range of each graph as an inequality.

3. 

4. 

Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time. What is the domain and range of the function in the graph?

5. 

6. 

Find the domain of each function

7. $f(x) = 3\sqrt{x-2}$

8. $f(x) = 5\sqrt{x+3}$

9. $f(x) = 3 - \sqrt{6-2x}$

10. $f(x) = 5 - \sqrt{10-2x}$

11. $f(x) = \dfrac{9}{x-6}$

12. $f(x) = \dfrac{6}{x-8}$

13. $f(x) = \dfrac{3x+1}{4x+2}$

14. $f(x) = \dfrac{5x+3}{4x-1}$

15. $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

16. $f(x) = \dfrac{\sqrt{x+5}}{x-6}$

17. $f(x) = \dfrac{x-3}{x^2+9x-22}$

18. $f(x) = \dfrac{x-8}{x^2+8x-9}$

Given each function, evaluate: $f(-1)$, $f(0)$, $f(2)$, $f(4)$

19. $f(x) = \begin{cases} 7x+3 & if \quad x < 0 \\ 7x+6 & if \quad x \geq 0 \end{cases}$

20. $f(x) = \begin{cases} 4x-9 & if \quad x < 0 \\ 4x-18 & if \quad x \geq 0 \end{cases}$

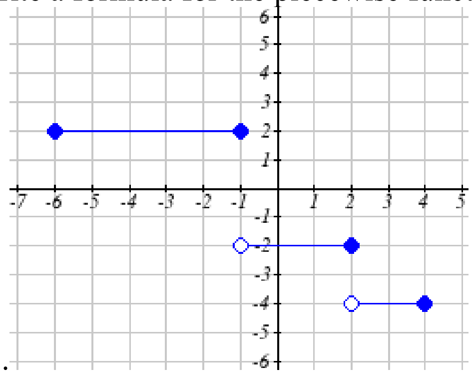
21. $f(x) = \begin{cases} x^2-2 & if \quad x < 2 \\ 4+|x-5| & if \quad x \geq 2 \end{cases}$

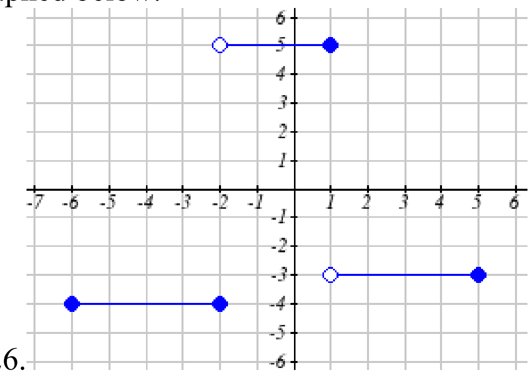
22. $f(x) = \begin{cases} 4-x^3 & if \quad x < 1 \\ \sqrt{x+1} & if \quad x \geq 1 \end{cases}$

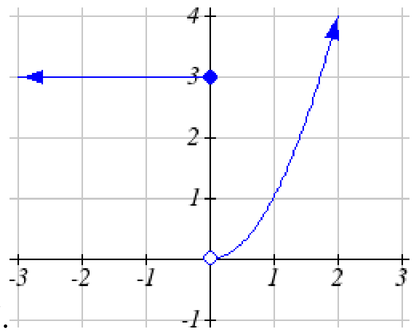
23. $f(x) = \begin{cases} 5x & if \quad x < 0 \\ 3 & if \quad 0 \leq x \leq 3 \\ x^2 & if \quad x > 3 \end{cases}$

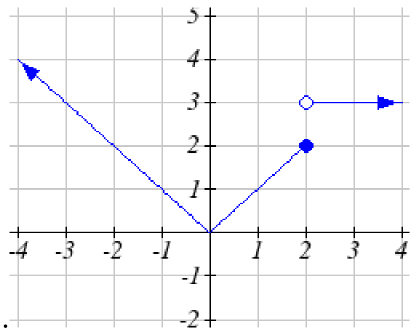
24. $f(x) = \begin{cases} x^3+1 & if \quad x < 0 \\ 4 & if \quad 0 \leq x \leq 3 \\ 3x+1 & if \quad x > 3 \end{cases}$

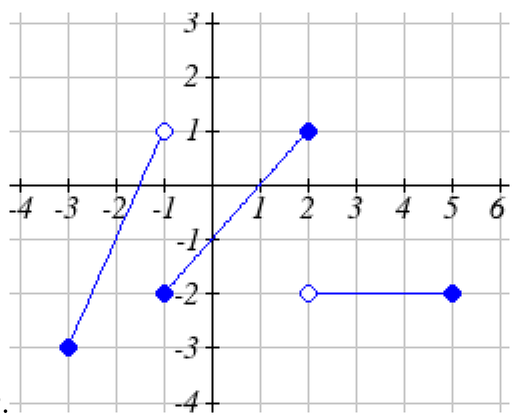
Write a formula for the piecewise function graphed below.

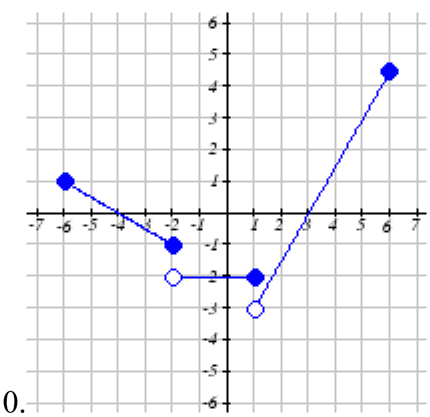
25. 

26. 

27. 

28. 

29. 

30. 

Sketch a graph of each piecewise function

31. $f(x)=\begin{cases} |x| & if \quad x<2 \\ 5 & if \quad x\ge 2 \end{cases}$

32. $f(x)=\begin{cases} 4 & if \quad x<0 \\ \sqrt{x} & if \quad x\ge 0 \end{cases}$

33. $f(x)=\begin{cases} x^2 & if \quad x<0 \\ x+2 & if \quad x\ge 0 \end{cases}$

34. $f(x)=\begin{cases} x+1 & if \quad x<1 \\ x^3 & if \quad x\ge 1 \end{cases}$

35. $f(x)=\begin{cases} 3 & if \quad x\le -2 \\ -x+1 & if \quad -2<x\le 1 \\ 3 & if \quad x>1 \end{cases}$

36. $f(x)=\begin{cases} -3 & if \quad x\le -2 \\ x-1 & if \quad -2<x\le 2 \\ 0 & if \quad x>2 \end{cases}$

Section 1.3 Rates of Change and Behavior of Graphs

Since functions represent how the output quantity varies with the input quantity, it is natural to ask how the values of the function are changing.

For example, the function *C(t)* below gives the average cost, in dollars, of a gallon of gasoline *t* years after 2000.

t	2	3	4	5	6	7	8	9
C(t)	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from $1.47 to $2.14, an increase of $0.67. While this is interesting, it might be more useful to look at how much the price changed *each year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by $0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7\,years} \approx 0.096$ dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are "<u>output units</u> per <u>input units</u>"

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns $9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by $4,000 per quarter

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1

Using the cost of gas function from earlier, find the average rate of change between 2007 and 2009

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50.

The average rate of change is then $\frac{-\$0.50}{2\,years}$ = -0.25 dollars per year

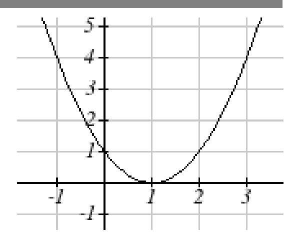
Try it Now

1. Using the same cost of gas function, find the average rate of change between 2003 and 2008

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 2

Given the function *g(t)* shown here, find the average rate of change on the interval [0, 3].



At t = 0, the graph shows $g(0)=1$
At t = 3, the graph shows $g(3)=4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0}=\frac{3}{3}=1$$

Example 3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
D(t) (miles)	10	55	90	153	214	240	292	300

Here, your average speed is the average rate of change.
You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0}=\frac{282}{6}=47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation

Average Rate of Change using Function Notation

Given a function *f(x)*, the average rate of change on the interval [*a*, *b*] is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b) - f(a)}{b - a}$$

Example 4

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4]

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

Try it Now

2. Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9]

Example 5

The magnetic force *F*, measured in Newtons, between two magnets is related to the distance between the magnets *d*, in centimeters, but the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6]

$$\text{Average rate of change} = \frac{F(6) - F(2)}{6 - 2}$$ Evaluating the function

$\dfrac{F(6)-F(2)}{6-2}=$

$\dfrac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2}$ Simplifying

$\dfrac{\frac{2}{36}-\frac{2}{4}}{4}$ Combining the numerator terms

$\dfrac{\frac{-16}{36}}{4}$ Simplifying further

$\dfrac{-1}{9}$ Newtons per centimeter

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter. Equivalently, it tells us the magnetic force decreases, on average by 1 Newton for each 9 centimeters the distance increases.

Example 6

Find the average rate of change of $g(t)=t^2+3t+1$ on the interval $[0,a]$. Your answer will be an expression involving *a*.

Using the average rate of change formula

$\dfrac{g(a)-g(0)}{a-0}$ Evaluating the function

$\dfrac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0}$ Simplifying

$\dfrac{a^2+3a+1-1}{a}$ Simplifying further, and factoring

$\dfrac{a(a+3)}{a}$ Cancelling the common factor *a*

$a+3$

This result tells us the average rate of change between $t = 0$ and any other point $t = a$. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

Try it Now

3. Find the average rate of change of $f(x)=x^3+2$ on the interval $[a,a+h]$

Graphical Behavior of Functions

As part of exploring how functions change, it is interesting to explore the graphical behavior of functions.

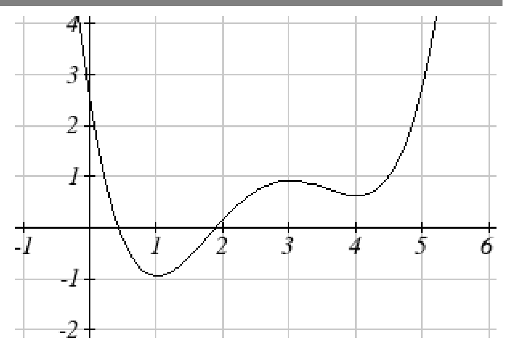
Increasing/Decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase. More formally, a function is increasing if $f(b) > f(a)$ for any two input values a and b in the interval with $b>a$. The average rate of change of an increasing function is **positive.**

A function is **decreasing** on an interval if the function values decrease as the inputs increase. More formally, a function is decreasing if $f(b) < f(a)$ for any two input values a and b in the interval with $b>a$. The average rate of change of a decreasing function is **negative.**

Example 7

Given the function $p(t)$ graphed here, on what intervals does the function appear to be increasing?



The function appears to be increasing from $t = 1$ to $t = 3$, and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1,3)$ and the interval $(4,\infty)$

Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is not technically increasing at $t = 1$, 3, or 4. At those points, the function is neither increasing nor decreasing.

Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**.

A point where a function changes from decreasing to increasing is called a **local minimum**.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

Example 8

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

t	2	3	4	5	6	7	8	9
C(t)	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

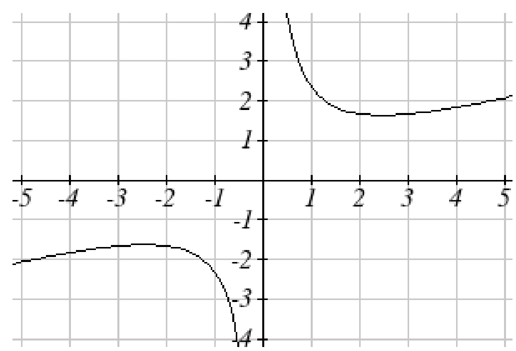
It appears that the cost of gas is increasing from $t = 2$ to $t = 8$. It appears the cost of gas decreased from $t = 8$ to $t = 9$, so the function appears to be decreasing on the interval (8, 9).

Since the function appears to change from increasing to decreasing at $t = 8$, there is local maximum at $t = 8$.

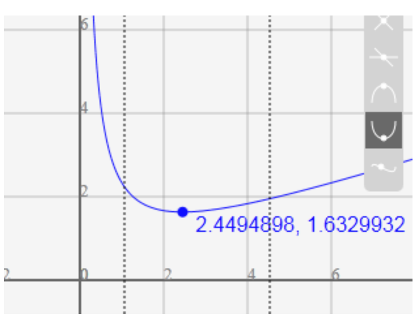
Example 9

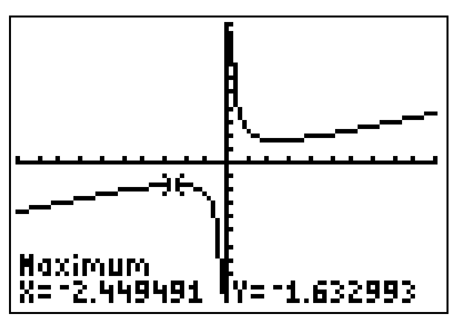
Use a graph to estimate the local extrema of the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Use these to determine the intervals on which the function is increasing.

Using technology to graph the function, it appears there is a local minimum somewhere between $x = 2$ and $x = 3$, and a symmetric local maximum somewhere between $x = -3$ and $x = -2$.



Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.





Based on these estimates, the function is increasing on the intervals $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that while we expect the extrema to be symmetric, the two different technologies are only the same up to 4 decimals due to the approximation approach used by each.

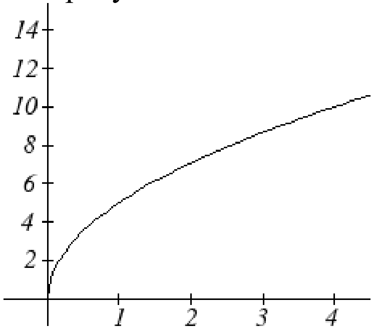
Try it Now

4. Use a graph of the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing.

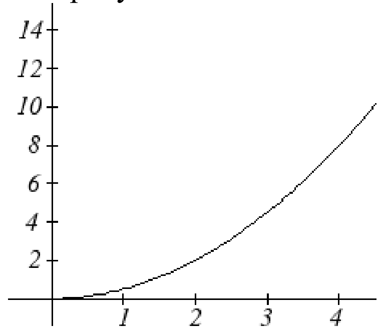
Concavity

The total sales, in thousands of dollars, for two companies over 4 weeks are shown.

Company A



Company B



As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can look how the average rate of change varies over different intervals. Using tables of values,

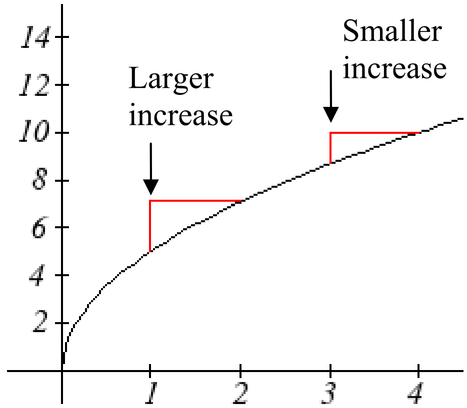
Company A

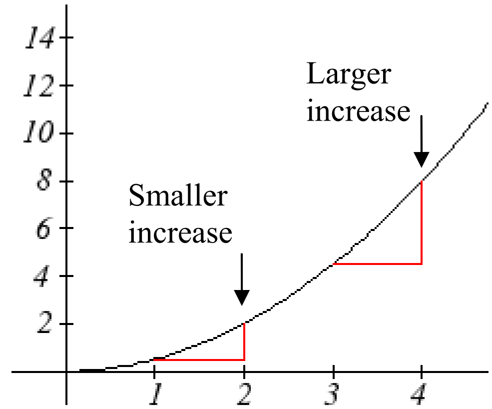
Week	Sales	Rate of Change
0	0	
		5
1	5	
		2.1
2	7.1	
		1.6
3	8.7	
		1.3
4	10	

Company B

Week	Sales	Rate of Change
0	0	
		0.5
1	0.5	
		1.5
2	2	
		2.5
3	4.5	
		3.5
4	8	

From the tables, we can see that the rate of change for company A is *decreasing*, while the rate of change for company B is *increasing*.





When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.

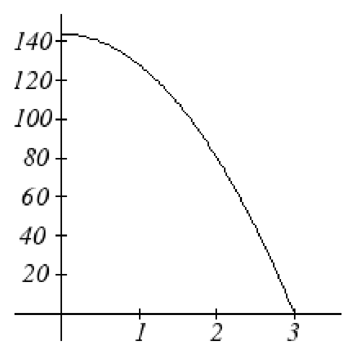
Concavity

A function is **concave up** if the rate of change is increasing.
A function is **concave down** if the rate of change is decreasing.
A point where a function changes from concave up to concave down or vice versa is called an **inflection point**.

Example 10

An object is thrown from the top of a building. The object's height in feet above ground after *t* seconds is given by the function $h(t) = 144 - 16t^2$ for $0 \le t \le 3$. Describe the concavity of the graph.

Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior



t	*h(t)*	Rate of Change
0	144	
		-16
1	128	
		-48
2	80	
		-80
3	0	

Notice that the rates of change are becoming more negative, so the rates of change are *decreasing*. This means the function is concave down.

Example 11

The value, V, of a car after t years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down?

t	0	2	4	6	8
$V(t)$	28000	24342	21162	18397	15994

Since the values are getting smaller, we can determine that the value is decreasing. We can compute rates of change to determine concavity.

t	0	2	4	6	8
$V(t)$	28000	24342	21162	18397	15994
Rate of change		-1829	-1590	-1382.5	-1201.5

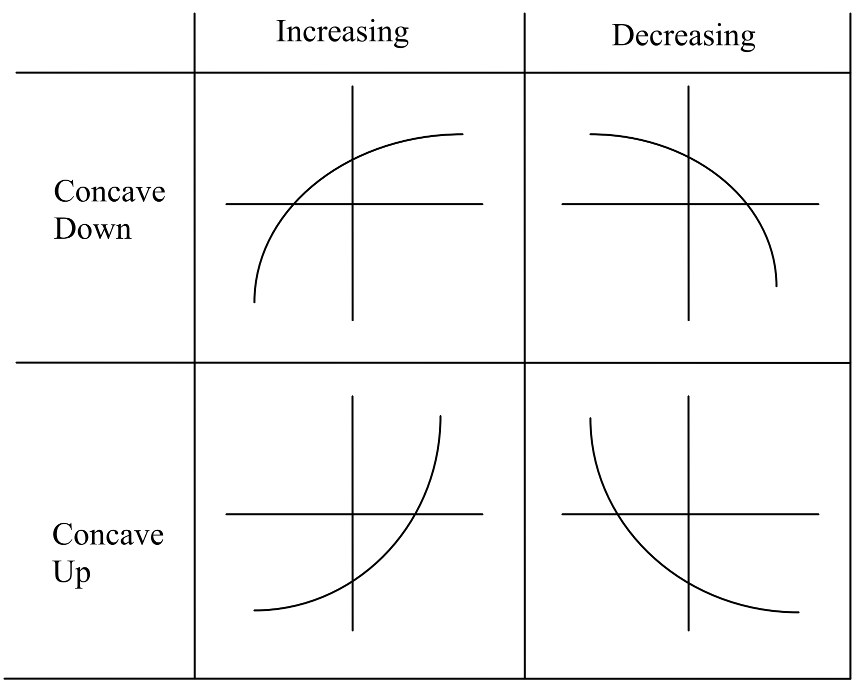
Since these values are becoming less negative, the rates of change are *increasing*, so this function is concave up.

Try it Now

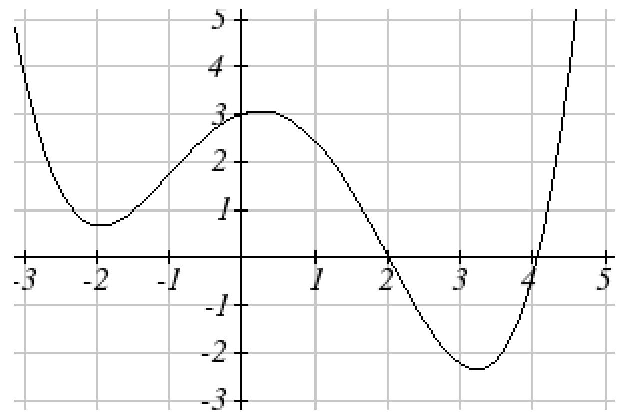
5. Is the function described in the table below concave up or concave down?

x	0	5	10	15	20
$g(x)$	10000	9000	7000	4000	0

Graphically, concave down functions bend downwards like a frown, and concave up function bend upwards like a smile.



Example 12



Estimate from the graph shown the intervals on which the function is concave down and concave up.

On the far left, the graph is decreasing but concave up, since it is bending upwards. It begins increasing at $x = -2$, but it continues to bend upwards until about $x = -1$.

From $x = -1$ the graph starts to bend downward, and continues to until about $x = 2$. The graph then begins curving upwards for the remainder of the graph shown.

From this, we can estimate that the graph is concave up on the intervals $(-\infty,-1)$ and $(2,\infty)$, and is concave down on the interval $(-1,2)$. The graph has inflection points at $x = -1$ and $x = 2$.

Try it Now

6. Using the graph from Try it Now 4, $f(x) = x^3 - 6x^2 - 15x + 20$, estimate the intervals on which the function is concave up and concave down.

Behaviors of the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

Function	Increasing/Decreasing	Concavity
Constant Function $f(x) = c$	Neither increasing nor decreasing	Neither concave up nor down
Identity Function $f(x) = x$	Increasing	Neither concave up nor down
Quadratic Function $f(x) = x^2$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$ Minimum at $x = 0$	Concave up $(-\infty,\infty)$
Cubic Function $f(x) = x^3$	Increasing	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$ Inflection point at (0,0)
Reciprocal odd $f(x) = \frac{1}{x}$	Decreasing $(-\infty,0) \cup (0,\infty)$	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$

Function	Increasing/Decreasing	Concavity
Reciprocal even $f(x)=\frac{1}{x^2}$	Increasing on $(-\infty,0)$ Decreasing on $(0,\infty)$	Concave up on $(-\infty,0)\cup(0,\infty)$
Cube Root $f(x)=\sqrt[3]{x}$	Increasing	Concave down on $(0,\infty)$ Concave up on $(-\infty,0)$ Inflection point at (0,0)
Square Root $f(x)=\sqrt{x}$	Increasing on $(0,\infty)$	Concave down on $(0,\infty)$
Absolute Value $f(x)=\lvert x\rvert$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$	Neither concave up or down

Important Topics of This Section

Rate of Change
Average Rate of Change
Calculating Average Rate of Change using Function Notation
Increasing/Decreasing
Local Maxima and Minima (Extrema)
Inflection points
Concavity

Try it Now Answers

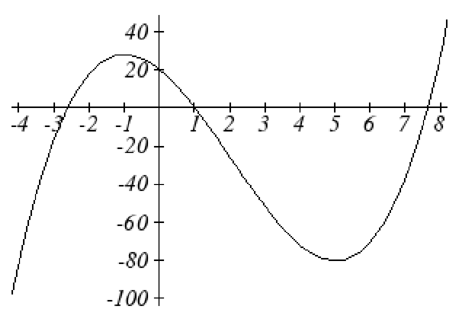
1. $\frac{\$3.01-\$1.69}{5\,years}=\frac{\$1.32}{5\,years}=$ 0.264 dollars per year.

2. Average rate of change $=\frac{f(9)-f(1)}{9-1}=\frac{\left(9-2\sqrt{9}\right)-\left(1-2\sqrt{1}\right)}{9-1}=\frac{(3)-(-1)}{9-1}=\frac{4}{8}=\frac{1}{2}$

3. $\frac{f(a+h)-f(a)}{(a+h)-a}=\frac{\left((a+h)^3+2\right)-\left(a^3+2\right)}{h}=\frac{a^3+3a^2h+3ah^2+h^3+2-a^3-2}{h}=$

$\frac{3a^2h+3ah^2+h^3}{h}=\frac{h\left(3a^2+3ah+h^2\right)}{h}=3a^2+3ah+h^2$

4. Based on the graph, the local maximum appears to occur at (-1, 28), and the local minimum occurs at (5,-80) The function is increasing on $(-\infty,-1)\cup(5,\infty)$ and decreasing on $(-1,5)$



5. Calculating the rates of change, we see the rates of change become *more* negative, so the rates of change are *decreasing*. This function is concave down.

x	0	5	10	15	20
$g(x)$	10000	9000	7000	4000	0
Rate of change	-1000	-2000	-3000	-4000	

6. Looking at the graph, it appears the function is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$

Section 1.3 Exercises

1. The table below gives the annual sales (in millions of dollars) of a product. What was the average rate of change of annual sales…
a) Between 2001 and 2002 b) Between 2001 and 2004

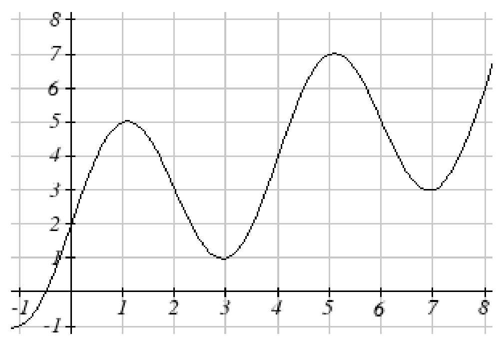
year	1998	1999	2000	2001	2002	2003	2004	2005	2006
sales	201	219	233	243	249	251	249	243	233

2. The table below gives the population of a town, in thousands. What was the average rate of change of population…
a) Between 2002 and 2004 b) Between 2002 and 2006

year	2000	2001	2002	2003	2004	2005	2006	2007	2008
population	87	84	83	80	77	76	75	78	81

3. Based on the graph shown, estimate the average rate of change from $x = 1$ to $x = 4$.

4. Based on the graph shown, estimate the average rate of change from $x = 2$ to $x = 5$.



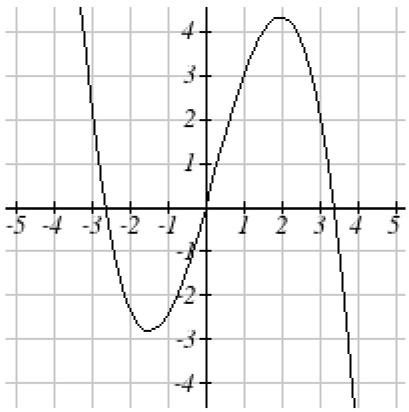
Find the average rate of change of each function on the interval specified.

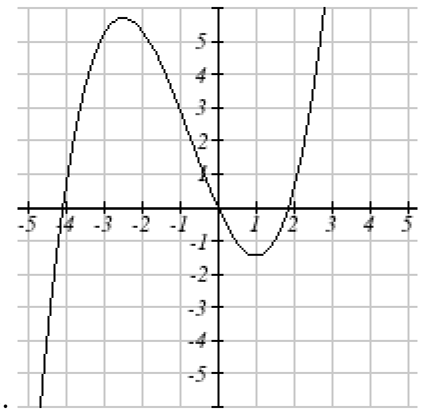
5. $f(x) = x^2$ on [1, 5]
6. $q(x) = x^3$ on [-4, 2]
7. $g(x) = 3x^3 - 1$ on [-3, 3]
8. $h(x) = 5 - 2x^2$ on [-2, 4]
9. $k(t) = 6t^2 + \frac{4}{t^3}$ on [-1, 3]
10. $p(t) = \frac{t^2 - 4x + 1}{t^2 + 3}$ on [-3, 1]

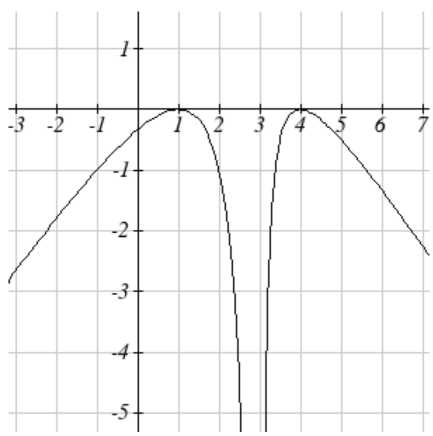
Find the average rate of change of each function on the interval specified. Your answers will be expressions.

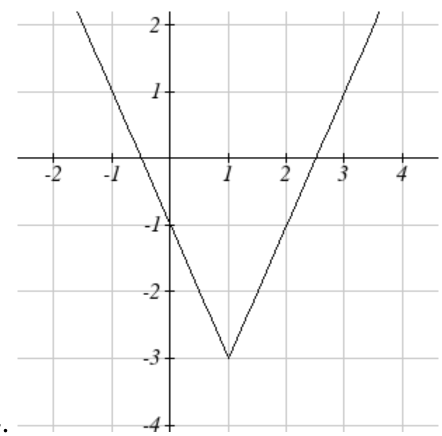
11. $f(x) = 4x^2 - 7$ on [1, b]
12. $g(x) = 2x^2 - 9$ on [4, b]
13. $h(x) = 3x + 4$ on [2, 2+h]
14. $k(x) = 4x - 2$ on [3, 3+h]
15. $a(t) = \frac{1}{t+4}$ on [9, 9+h]
16. $b(x) = \frac{1}{x+3}$ on [1, 1+h]
17. $j(x) = 3x^3$ on [1, 1+h]
18. $r(t) = 4t^3$ on [2, 2+h]
19. $f(x) = 2x^2 + 1$ on [x, x+h]
20. $g(x) = 3x^2 - 2$ on [x, x+h]

For each function graphed, estimate the intervals on which the function is increasing and decreasing.

21. 

22. 

23. 

24. 

For each table below, select whether the table represents a function that is increasing or decreasing, and whether the function is concave up or concave down.

25.

x	*f(x)*
1	2
2	4
3	8
4	16
5	32

26.

x	*g(x)*
1	90
2	70
3	80
4	75
5	72

27.

x	*h(x)*
1	300
2	290
3	270
4	240
5	200

28.

x	*k(x)*
1	0
2	15
3	25
4	32
5	35

29.

x	*f(x)*
1	-10
2	-25
3	-37
4	-47
5	-54

30.

x	*g(x)*
1	-200
2	-190
3	-160
4	-100
5	0

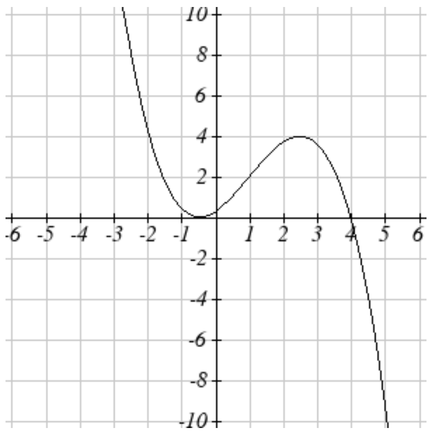
31.

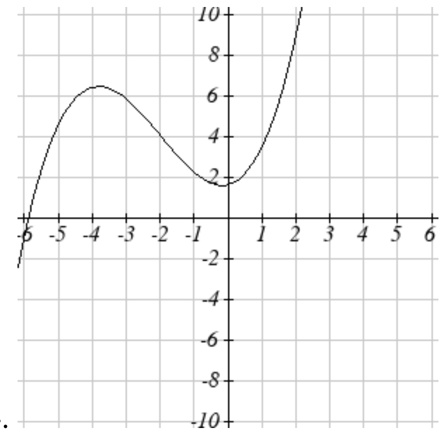
x	*h(x)*
1	-100
2	-50
3	-25
4	-10
5	0

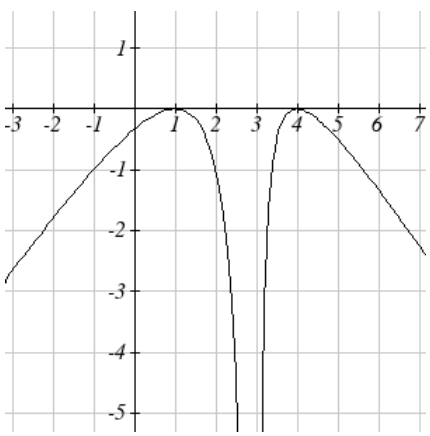
32.

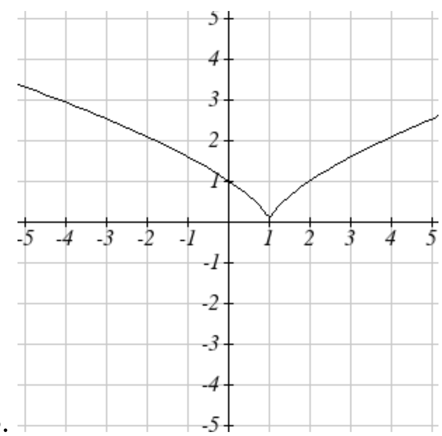
x	*k(x)*
1	-50
2	-100
3	-200
4	-400
5	-900

For each function graphed, estimate the intervals on which the function is concave up and concave down, and the location of any inflection points.

33. 

34. 

35. 

36. 

Use a graph to estimate the local extrema and inflection points of each function, and to estimate the intervals on which the function is increasing, decreasing, concave up, and concave down.

37. $f(x) = x^4 - 4x^3 + 5$

38. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

39. $g(t) = t\sqrt{t+3}$

40. $k(t) = 3t^{2/3} - t$

41. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

42. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Section 1.4 Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the Cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, $C(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Celsius, and the second, $T(d)$, gives the average daily temperature of a particular city on day d of the year. If we wanted to determine the cost of heating the house on the 5th day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function $T(d)$, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. We could then use that temperature as the input to the $C(T)$ function to find the cost to heat the house on the 5th day of the year: $C(T(5))$.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write $f(g(x))$, and read this as "f of g of x" or "f composed with g at x".

An alternate notation for composition uses the composition operator: $\circ$
$(f \circ g)(x)$ is read "f of g of x" or "f composed with g of x", just like $f(g(x))$

Example 1

Suppose $c(s)$ gives the number of calories burned doing s sit-ups, and $s(t)$ gives the number of sit-ups a person can do in t minutes. Interpret $c(s(3))$.

When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside function in the composition is $s(3)$. Since the input to the s function is time, the 3 is representing 3 minutes, and $s(3)$ is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the $c(s)$ function will gives us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it *is* essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 2

Suppose *f(x)* gives miles that can be driven in *x* hours, and *g(y)* gives the gallons of gas used in driving *y* miles. Which of these expressions is meaningful: *f(g(y))* or *g(f(x))*?

The expression *g(y)* takes miles as the input and outputs a number of gallons. The function *f(x)* is expecting a number of hours as the input; trying to give it a number of gallons as input does not make sense. Remember the units have to match, and number of gallons does not match number of hours and so the expression *f(g(y))* is meaningless.

The expression *f(x)* takes hours as input and outputs a number of miles driven. The function *g(y)* is expecting a number of miles as the input, so giving the output of the *f(x)* function (miles driven) as an input value for g(*y*), where gallons of gas depends on miles driven, does make sense. The expression *g(f(x))* makes sense, and will give the number of gallons of gas used, *g*, driving a certain number of miles, *f(x)*, in x hours.

Try it Now

1. In a department store you see a sign that says 50% off of clearance merchandise, so final cost *C* depends on the clearance price, *p*, according to the function *C(p)*. Clearance price, *p* depends on the original discount, *d*, given to the clearance item, *p(d)*. Interpret *C(p(d))*.

Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph, as discussed in section 1.1. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 3

Using the tables below, evaluate $f(g(3))$ and $g(f(4))$

x	*f(x)*
1	6
2	8
3	3
4	1

x	*g(x)*
1	3
2	5
3	2
4	7

To evaluate $f(g(3))$, we start from the inside with the value 3. We then evaluate the inside function $g(3)$ using the table that defines the function *g*: $g(3) = 2$. We can then use that result as the input to the *f* function, so $g(3)$ is replaced by the equivalent value 2 and we get $f(2)$. Then using the table that defines the function *f*, we find that $f(2) = 8$.

$f(g(3)) = f(2) = 8$

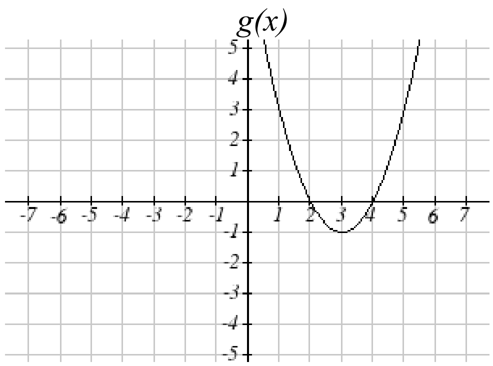
To evaluate $g(f(4))$, we first evaluate the inside $f(4)$ using the first table: $f(4) = 1$.
Then using the table for g we can evaluate:
$g(f(4)) = g(1) = 3$

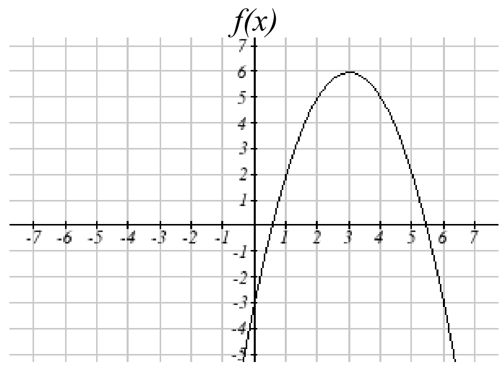
Try it Now

2. Using the tables from the example above, evaluate $f(g(1))$ and $g(f(3))$

Example 4

Using the graphs below, evaluate $f(g(1))$





To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the *g(x)* function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1) = 3$. Using this value as the input to the *f* function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the *f(x)* function, and finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Try it Now

3. Using the graphs from the previous example, evaluate $g(f(2))$.

Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Since the inside evaluation is $h(1)$ we start by evaluating the *h(x)* function at 1:
$h(1) = 3(1) + 2 = 5$

Then $f(h(1)) = f(5)$, so we evaluate the *f(t)* function at an input of 5:
$f(h(1)) = f(5) = 5^2 - 5 = 20$

Try it Now

4. Using the functions from the example above, evaluate $h(f(-2))$

While we can compose the functions as above for each individual input value, sometimes it would be really helpful to find a single formula which will calculate the result of a composition *f(g(x))*. To do this, we will extend our idea of function evaluation. Recall that when we evaluate a function like $f(t) = t^2 - t$, we put whatever value is inside the parentheses after the function name into the formula wherever we see the input variable.

Example 6

Given $f(t) = t^2 - t$, evaluate $f(3)$ and $f(-2)$

$f(3) = 3^2 - 3$
$f(-2) = (-2)^2 - (-2)$

We could simplify the results above if we wanted to
$f(3) = 3^2 - 3 = 9 - 3 = 6$
$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

We are not limited, however, to putting a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an entire equation, provided we put the input expression everywhere we see the input variable.

Example 7

Using the function from the previous example, evaluate *f(a)*

This means that the input value for *t* is some unknown quantity *a*. As before, we evaluate by replacing the input variable *t* with the input quantity, in this case *a*.
$f(a) = a^2 - a$

The same idea can then be applied to expressions more complicated than a single letter.

Example 8

Using the same *f(t)* function from above, evaluate $f(x+2)$

Everywhere in the formula for *f* where there was a *t*, we would replace it with the input $(x+2)$. Since in the original formula the input *t* was squared in the first term, the entire input $x+2$ needs to be squared when we substitute, so we need to use grouping parentheses. To avoid problems, it is advisable to always insert input with parentheses.

$$f(x+2)=(x+2)^2-(x+2)$$

We could simplify this expression further to $f(x+2)=x^2+3x+2$ if we wanted to

$f(x+2)=(x+2)(x+2)-(x+2)$ Use the "FOIL" technique (first, outside, inside, last)

$f(x+2)=x^2+2x+2x+4-(x+2)$ distribute the negative sign

$f(x+2)=x^2+2x+2x+4-x-2$ combine like terms

$f(x+2)=x^2+3x+2$

Example 9

Using the same function, evaluate $f(t^3)$

Note that in this example, the same variable is used in the input expression and as the input variable of the function. This doesn't matter – we still replace the original input *t* in the formula with the new input expression, t^3.

$$f(t^3)=(t^3)^2-(t^3)=t^6-t^3$$

Try it Now

5. Given $g(x)=3x-\sqrt{x}$, evaluate $g(t-2)$

This now allows us to find an expression for a composition of functions. If we want to find a formula for *f(g(x))*, we can start by writing out the formula for *g(x)*. We can then evaluate the function *f(x)* at that expression, as in the examples above.

Example 10

Let $f(x)=x^2$ and $g(x)=\frac{1}{x}-2x$, find *f(g(x))* and *g(f(x))*

To find *f(g(x))*, we start by evaluating the inside, writing out the formula for *g(x)*

$$g(x)=\frac{1}{x}-2x$$

We then use the expression $\left(\frac{1}{x}-2x\right)$ as input for the function *f*.

$$f(g(x))=f\left(\frac{1}{x}-2x\right)$$

We then evaluate the function *f(x)* using the formula for *g(x)* as the input.

Since $f(x)=x^2$ then $f\left(\frac{1}{x}-2x\right)=\left(\frac{1}{x}-2x\right)^2$

This gives us the formula for the composition: $f(g(x))=\left(\frac{1}{x}-2x\right)^2$

Likewise, to find *g(f(x))*, we evaluate the inside, writing out the formula for *f(x)*

$$g(f(x))=g\left(x^2\right)$$

Now we evaluate the function *g(x)* using x^2 as the input.

$$g(f(x))=\frac{1}{x^2}-2x^2$$

Try it Now

6. Let $f(x)=x^3+3x$ and $g(x)=\sqrt{x}$, find *f(g(x))* and *g(f(x))*

Example 11

A city manager determines that the tax revenue, *R*, in millions of dollars collected on a population of *p* thousand people is given by the formula $R(p)=0.03p+\sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t)=60+2t+0.3t^2$, where *t* is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have to predict the city population, and then use that result as the input to the tax function. So we need to find *R(p(t))*. Evaluating this,

$$R(p(t)) = R\left(60 + 2t + 0.3t^2\right) = 0.03\left(60 + 2t + 0.3t^2\right) + \sqrt{60 + 2t + 0.3t^2}$$

This composition gives us a single formula which can be used to predict the tax revenue given the year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when $t = 7$ (because t is measured in years after 2010)

$$R(p(7)) = 0.03\left(60 + 2(7) + 0.3(7)^2\right) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars}$$

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions.

Example 12

Write $f(x) = 3 + \sqrt{5 - x^2}$ as the composition of two functions.

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for *f(x)*. As one possibility, we might notice that $5 - x^2$ is the inside of the square root. We could then decompose the function as:

$h(x) = 5 - x^2$

$g(x) = 3 + \sqrt{x}$

We can check our answer by recomposing the functions:

$g(h(x)) = g\left(5 - x^2\right) = 3 + \sqrt{5 - x^2}$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be $h(x) = x^2$ and $g(x) = 3 + \sqrt{5 - x}$

Important Topics of this Section

Definition of Composition of functions

Compositions using:

- Words
- Tables
- Graphs
- Equations

Try it Now Answers

1. The final cost, *C*, depends on the clearance price, *p*, which is based on the original discount, *d*. (or the original discount *d*, determines the clearance price and the final cost is half of the clearance price)
2. $f(g(1)) = f(3) = 3$ and $g(f(3)) = g(3) = 2$
3. $g(f(2)) = g(5) = 3$
4. $h(f(-2)) = h(6) = 20$ *did you remember to insert your input values using parenthesis?*
5. $g(t-2) = 3(t-2) - \sqrt{(t-2)}$
6. $f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^3 + 3\left(\sqrt{x}\right)$

   $g(f(x)) = g\left(x^3 + 3x\right) = \sqrt{\left(x^3 + 3x\right)}$

Section 1.4 Exercises

Given each pair of equations, calculate $f(g(0))$ and $g(f(0))$

1. $f(x)=4x+8$, $g(x)=7-x^2$
2. $f(x)=5x+7$, $g(x)=4-2x^2$
3. $f(x)=\sqrt{x+4}$, $g(x)=12-x^3$
4. $f(x)=\dfrac{1}{x+2}$, $g(x)=4x+3$

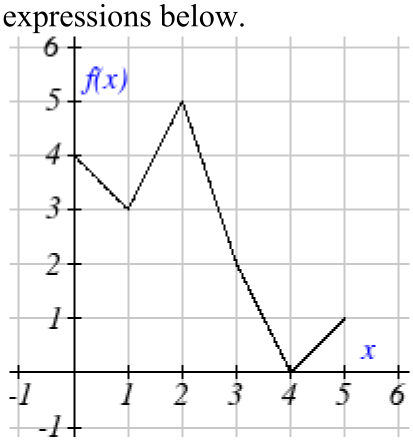
Use the table of values to evaluate each expression

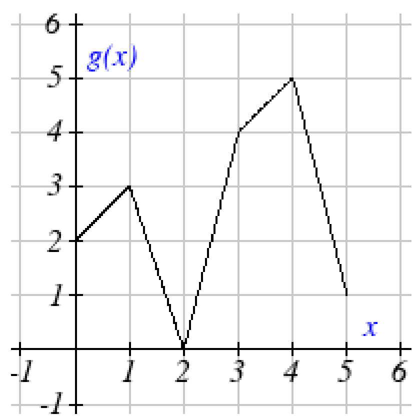
5. $f(g(8))$
6. $f(g(5))$
7. $g(f(5))$
8. $g(f(3))$
9. $f(f(4))$
10. $f(f(1))$
11. $g(g(2))$
12. $g(g(6))$

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Use the graphs to evaluate the expressions below.

13. $f(g(3))$
14. $f(g(1))$
15. $g(f(1))$
16. $g(f(0))$
17. $f(f(5))$
18. $f(f(4))$
19. $g(g(2))$
20. $g(g(0))$





For each pair of functions, find $f(g(x))$ and $g(f(x))$. Simplify your answers.

21. $f(x)=\dfrac{1}{x-6}$, $g(x)=\dfrac{7}{x}+6$
22. $f(x)=\dfrac{1}{x-4}$, $g(x)=\dfrac{2}{x}+4$
23. $f(x)=x^2+1$, $g(x)=\sqrt{x+2}$
24. $f(x)=\sqrt{x}+2$, $g(x)=x^2+3$

25. $f(x)=|x|$, $g(x)=5x+1$

26. $f(x)=\sqrt[3]{x}$, $g(x)=\dfrac{x+1}{x^3}$

27. If $f(x)=x^4+6$, $g(x)=x-6$ and $h(x)=\sqrt{x}$, find $f(g(h(x)))$

28. If $f(x)=x^2+1$, $g(x)=\dfrac{1}{x}$ and $h(x)=x+3$, find $f(g(h(x)))$

29. Given functions $p(x)=\dfrac{1}{\sqrt{x}}$ and $m(x)=x^2-4$, state the domains of the following functions using interval notation.
    a. Domain of $\dfrac{p(x)}{m(x)}$
    b. Domain of $p(m(x))$
    c. Domain of $m(p(x))$

30. Given functions $q(x)=\dfrac{1}{\sqrt{x}}$ and $h(x)=x^2-9$, state the domains of the following functions using interval notation.
    a. Domain of $\dfrac{q(x)}{h(x)}$
    b. Domain of $q(h(x))$
    c. Domain of $h(q(x))$

31. The function $D(p)$ gives the number of items that will be demanded when the price is *p*. The production cost, $C(x)$ is the cost of producing *x* items. To determine the cost of production when the price is \$6, you would do which of the following:
    a. Evaluate $D(C(6))$
    b. Evaluate $C(D(6))$
    c. Solve $D(C(x))=6$
    d. Solve $C(D(p))=6$

32. The function $A(d)$ gives the pain level on a scale of 0-10 experienced by a patient with *d* milligrams of a pain reduction drug in their system. The milligrams of drug in the patient's system after *t* minutes is modeled by $m(t)$. To determine when the patient will be at a pain level of 4, you would need to:
    a. Evaluate $A(m(4))$
    b. Evaluate $m(A(4))$
    c. Solve $A(m(t))=4$
    d. Solve $m(A(d))=4$

33. The radius *r*, in inches, of a balloon is related to the volume, *V*, by $r(V)=\sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after *t* seconds is given by $V(t)=10+20t$
    a. Find the composite function $r(V(t))$
    b. Find the time when the radius reaches 10 inches.

34. The number of bacteria in a refrigerated food product is given by $N(T)=23T^2-56T+1$, $3<T<33$ where *T* is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t)=5t+1.5$, where *t* is the time in hours.
    a. Find the composite function $N(T(t))$
    b. Find the time when the bacteria count reaches 6752

Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$

35. $h(x)=(x+2)^2$
36. $h(x)=(x-5)^3$
37. $h(x)=\dfrac{3}{x-5}$
38. $h(x)=\dfrac{4}{(x+2)^2}$
39. $h(x)=3+\sqrt{x-2}$
40. $h(x)=4+\sqrt[3]{x}$

41. Let $f(x)$ be a linear function, having form $f(x)=ax+b$ for constants *a* and *b*. [UW]
a. Show that $f(f(x))$ is a linear function
b. Find a function g(x) such that $\text{g}(\text{g}(\text{x}))=6\text{x}-8$

42. Let $f(x)=\frac{1}{2}x+3$ [UW]
    a. Sketch the graphs of $f(x), f(f(x)), f(f(f(x)))$ on the interval $-2\le x\le 10$.
    b. Your graphs should all intersect at the point (6, 6). The value x = 6 is called a fixed point of the function *f(x)*since $f(6)=6$; that is, 6 is fixed - it doesn't move when *f* is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(...f(x)...)))$.
    c. Linear functions (with the exception of $f(x)=x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x)=x^2-2$.
    d. Give a quadratic function whose fixed points are $x=-2$ and $x=3$.

43. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function $C(m) = \frac{70m^2}{10+m^2}$. [UW]

a. Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?

b. Find a function $m = g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C(g(h))$; what does this function calculate?

c. Find a function $z = v(s)$ that converts mph s into ft/sec z. Write out the formula for the new function $v(C(m)$; what does this function calculate?

Section 1.5 Transformation of Functions

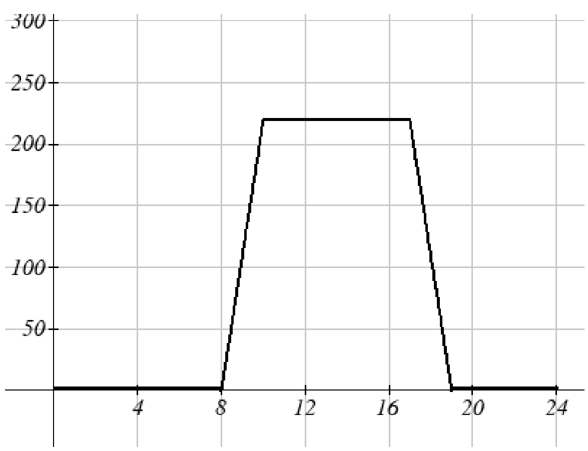
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs and equations in order to explain or solve it. When building models, it is often helpful to build off of existing formulas or models. Knowing the basic graphs of your tool-kit functions can help you solve problems by being able to model the behavior after something you already know. Unfortunately, the models and existing formulas we know are not always exactly the same as the ones presented in the problems we face.

Fortunately, there are systematic ways to shift, stretch, compress, flip and combine functions to help them become better models for the problems we are trying to solve. We can transform what we already know, into what we need, hence the name, “Transformation of functions.” When we have a story problem, formula, graph, or table, we can then transform that function in a variety of ways to form new equations.

Shifts

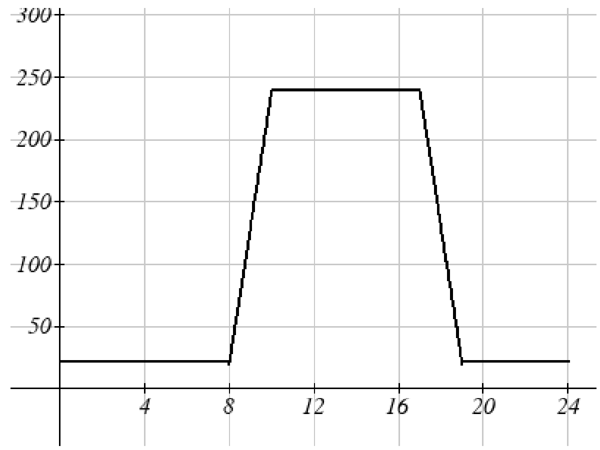
Example 1

To regulate temperature in our green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents V (in square feet) throughout the day, t in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.



We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up.

Notice that in the second graph, for the same input value, the output values have all increased by twenty, so if we call the new function *S(t)*, we could write $S(t) = V(t) + 20$.



Note that this notation tells us that for any value of *t*, *S(t)* can be found by evaluating the *V* function at the same input, then adding twenty to the result. This defines *S* as a transformation of the function *V*, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

Vertical Shift

Given a function *f(x)*, and if we define a new function *g(x)* as
$g(x) = f(x) + k$, where *k* is a constant
then *g(x)* is a **vertical shift** of the function *f(x)*, where all the output values have been increased by *k*.
If *k* is positive, then the graph will shift up
If *k* is negative, then the graph will shift down

Example 2

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x) - 3$

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the *g* function by subtracting 3 from the output values of the *f* function. For example,

$f(2) = 1$ is found from the given table
$g(x) = f(x) - 3$ is our given transformation
$g(2) = f(2) - 3 = 1 - 3 = -2$

Subtracting 3 from each *f(x)* value, we can complete a table of values for *g(x)*

x	2	4	6	8
g(x)	-2	0	4	8

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Try it Now

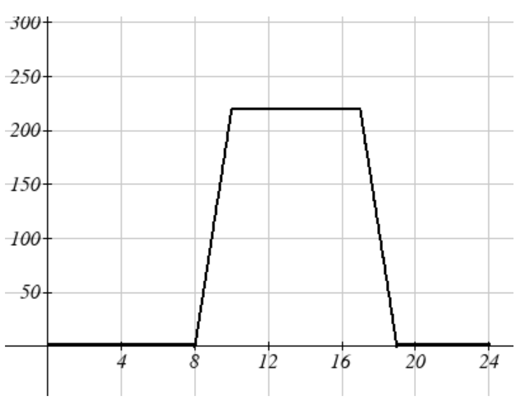
1. The function $h(t) = -4.9t^2 + 30t$ gives the height *h* of a ball (in meters) thrown upwards from the ground after *t* seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function *b(t)* to *h(t)*, then find a formula for *b(t)*.

The vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change the graph and meaning.

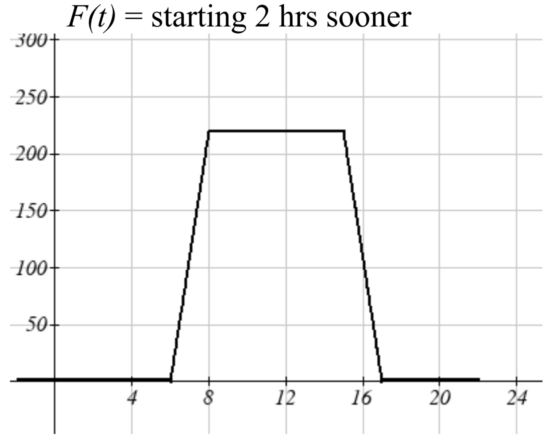
Example 3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.

V(t) = the original venting plan



F(t) = starting 2 hrs sooner



In the new graph, which we can call *F(t)*, at each time, the air flow is the same as the original function *V(t)* was two hours later. For example, in the original function *V*, the air flow starts to change at 8am, for the function *F(t)* the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$

Another example shows that the air flow first reached 220 at 10am in the original plan *V(t)* and in the new function *F(t)*, it first reaches 220 at 8am, so $V(10) = F(8)$.

In both cases we see that since *F(t)* starts 2 hours sooner, the same output values are reached when, $F(t) = V(t+2)$

Note that $V(t+2)$ had the affect of shifting the graph to the *left*.

Horizontal changes or “inside changes” affect the domain of a function (the input) instead of the range and often seem counter intuitive. The new function *F(t)* took away two hours from *V(t)* so to make them equal again, we have to compensate; we have to add 2 hrs to the input of *V* to get equivalent output values in *F*: $F(t) = V(t+2)$

Horizontal Shift

Given a function *f(x)*, and if we define a new function *g(x)* as
$g(x) = f(x+k)$, where *k* is a constant
then *g(x)* is a **horizontal shift** of the function *f(x)*
If *k* is positive, then the graph will shift left
If *k* is negative, then the graph will shift right

Example 4

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x-3)$

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = f(x-3)$ tells us that the output values of *g* are the same as the output value of *f* with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the *g* function, we will need an input value that is 3 *larger*: We input a value that is three larger for *g(x)* because the function takes three away before evaluating the function *f*.

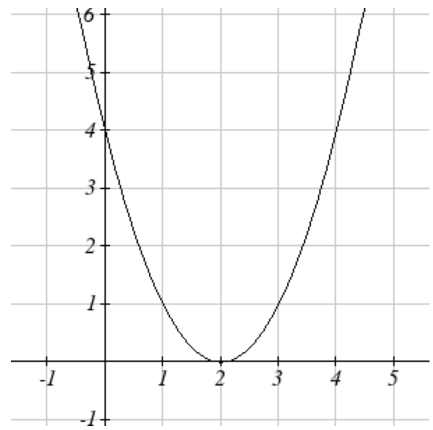
$g(5) = f(5-3) = f(2) = 1$

x	5	7	9	11
g(x)	1	3	7	11

The result is that the function *g(x)* has been shifted to the right by 3. Notice the output values for *g(x)* remain the same as the output values for *f(x)* in the chart, but the input values, *x*, have shifted to the right by 3; 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

Example 5

The graph to the right is a transformation of the toolkit function $f(x) = x^2$. Relate this new function *g(x)* to *f(x)*, and then find a formula for *g(x)*.



Notice that the graph looks almost identical in shape to the $f(x) = x^2$ function, but the *x* values are shifted to the right two units. The vertex used to be at (0, 0) but now the vertex is at (2, 0). The graph is the basic quadratic function shifted two to the right, so
$g(x) = f(x-2)$

Notice how we must input the value $x = 2$, to get the output value $y = 0$; the *x* values must be two units larger, because of the shift to the right by 2 units.

We can then use the definition of the *f(x)* function to write a formula for *g(x)* by evaluating $f(x-2)$:
Since $f(x) = x^2$ and $g(x) = f(x-2)$
$g(x) = f(x-2) = (x-2)^2$

If you find yourself having trouble determining whether the shift is +2 or -2, it might help to consider a single point on the graph. For a quadratic, looking at the bottommost point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the *f* function, we need to decide whether a +2 or -2 will work to satisfy $g(2) = f(2 ? 2) = f(0) = 0$. For this to work, we will need to subtract 2 from our input values.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

Example 6

The function *G(m)* gives the number of gallons of gas required to drive *m* miles. Interpret $G(m)+10$ and $G(m+10)$

$G(m)+10$ is adding 10 to the output, gallons. So this is 10 gallons of gas more than is required to drive *m* miles. So this is the gas required to drive *m* miles, plus another 10 gallons of gas.

$G(m+10)$ is adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 more than *m* miles.

Try it Now

2. Given the function $f(x)=\sqrt{x}$ graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$

a. Is this a horizontal or a vertical change?

b. Which way is the graph shifted and by how many units?

c. Graph *f(x)* and *g(x)* on the same axes

Now that we have two transformations, we can combine them together.

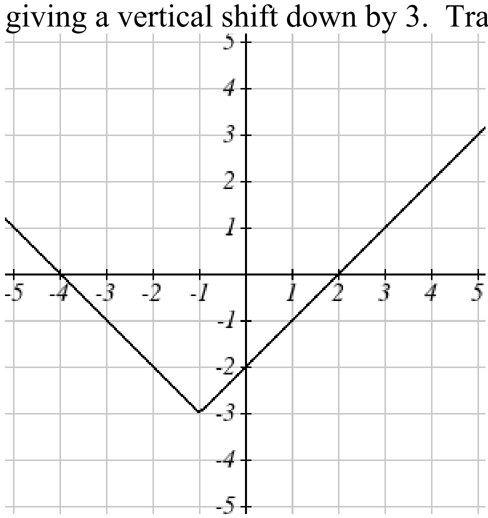
Remember:
Vertical Shifts are outside changes that affect the output (vertical) axis values shifting the transformed function up and down.

Horizontal Shifts are inside changes that affect the input (horizontal) axis values shifting the transformed function left and right.

Example 7

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives



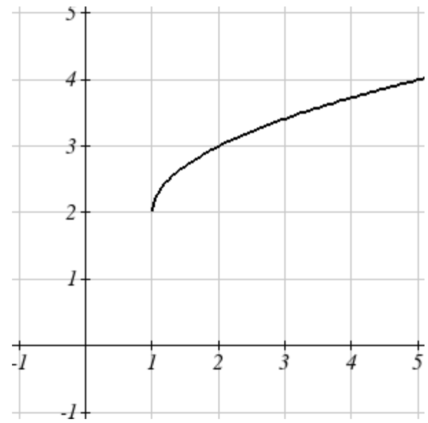
We could also find a formula for this transformation by evaluating the expression for *h(x)*:

$h(x)=f(x+1)-3$

$h(x)=|x+1|-3$

Example 8

Write a formula for the graph to the right, a transformation of the toolkit square root function.



The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as $h(x) = f(x-1)+2$. Using the formula for the square root function we can write

$h(x) = \sqrt{x-1}+2$

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

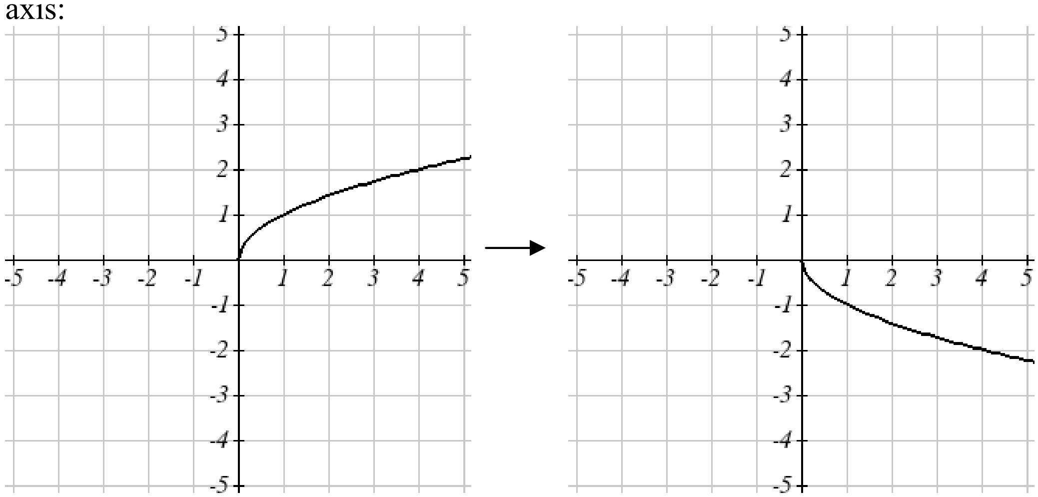
Reflections

Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

Example 9

Reflect the graph of $s(t) = \sqrt{t}$ both vertically and horizontally.

Reflecting the graph vertically, each output value will be reflected over the horizontal *t* axis:



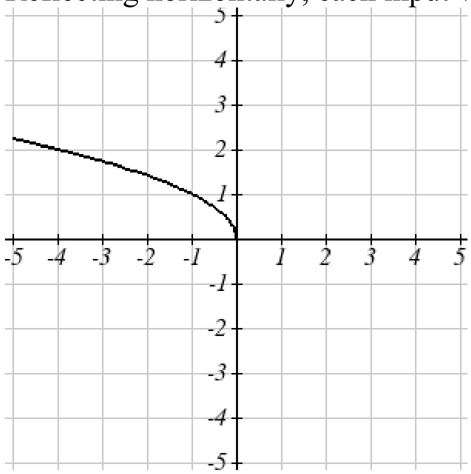
Since each output value is the opposite of the original output value, we can write

$V(t) = -s(t)$

$V(t) = -\sqrt{t}$

Notice this is an outside change or vertical change that affects the output *s(t)* values so the negative sign belongs outside of the function.

Reflecting horizontally, each input value will be reflected over the vertical axis:



Since each input value is the opposite of the original input value, we can write

$H(t) = s(-t)$

$H(t) = \sqrt{-t}$

Notice this is an inside change or horizontal change that affects the input values so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the *V(t)* function the range $(-\infty, 0]$, and the horizontal reflection gives the *H(t)* function the domain $(-\infty, 0]$.

Reflections

Given a function *f(x)*, and if we define a new function *g(x)* as

$g(x) = -f(x)$,

then *g(x)* is a **vertical reflection** of the function *f(x)*, sometimes called a reflection about the *x*-axis

If we define a new function *g(x)* as

$g(x) = f(-x)$,

then *g(x)* is a **horizontal reflection** of the function *f(x)*, sometimes called a reflection about the *y*-axis

Example 10

A function *f(x)* is given as a table below. Create a table for the function $g(x) = -f(x)$ and $h(x) = f(-x)$

x	2	4	6	8
f(x)	1	3	7	11

For *g(x)*, this is a vertical reflection, so the x values stay the same and each output value will be the opposite of the original output value:

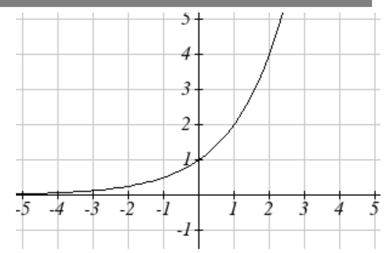
x	2	4	6	8
g(x)	-1	-3	-7	-11

For *h(x)*, this is a horizontal reflection, and each input value will be the opposite of the original input value and the h(x) values stay the same as the f(x) values:

x	-2	-4	-6	-8
h(x)	1	3	7	11

Example 11

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where *k* is the percentage of mastery that can be achieved after *t* practice sessions. This is a transformation of the function $f(t) = 2^{t}$ shown here. Sketch a graph of *k(t)*.

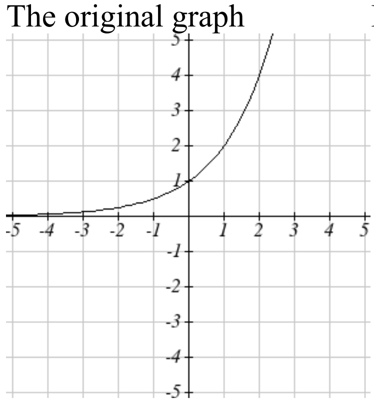


This equation combines three transformations into one equation.

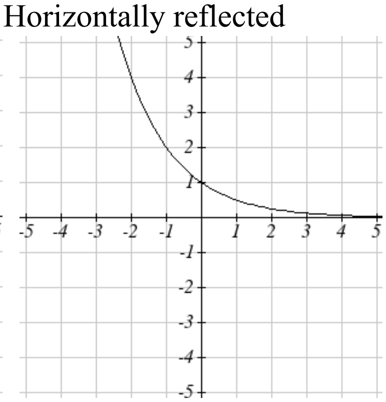
A horizontal reflection:	$f(-t) = 2^{-t}$	combined with
A vertical reflection:	$-f(-t) = -2^{-t}$	combined with
A vertical shift up 1:	$-f(-t) + 1 = -2^{-t} + 1$	

We can sketch a graph by applying these transformations one at a time to the original function:

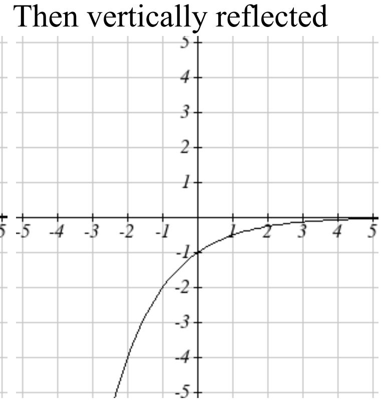
The original graph



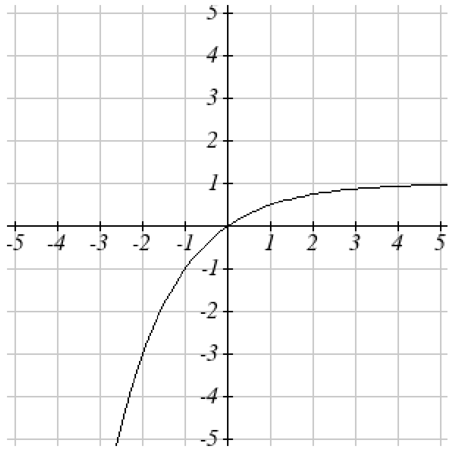
Horizontally reflected



Then vertically reflected



Then after shifting up 1, we get the final graph:



$k(t) = -f(-t) + 1 = -2^{-t} + 1$

Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0,1]$

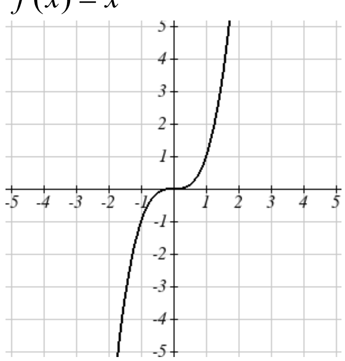
Try it Now

3. Given the toolkit function $f(x) = x^2$ graph *g(x) = -f(x)* and *h(x) = f(-x)*
Do you notice anything? Discuss your findings with a friend.

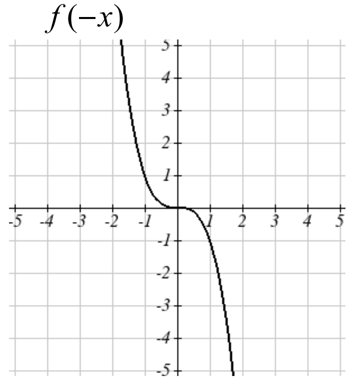
Some functions exhibit symmetry in which reflections result in the original graph. For example, reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We call these types of graphs symmetric about the *y*-axis.

Likewise, if the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph:

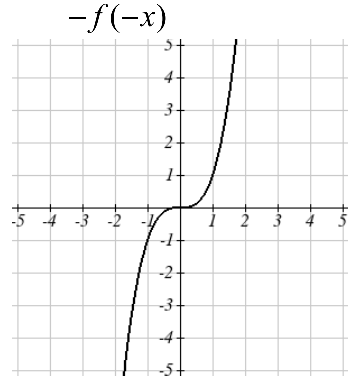
$f(x) = x^3$



$f(-x)$



$-f(-x)$



We call these graphs symmetric about the origin.

Even and Odd Functions

A function is called an **even function** if
$f(x) = f(-x)$
The graph of an even function is symmetric about the vertical axis

A function is called an **odd function** if
$f(x) = -f(-x)$
The graph of an odd function is symmetric about the origin

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, the $f(x) = 2^x$ function is neither even nor odd.

Example 12

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Without looking at a graph, we can determine this by finding formulas for the reflections, and seeing if they return us to the original function:

$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$
This does not return us to the original function, so this function is not even. We can now try also applying a horizontal reflection:

$-f(-x) = -\left(-x^3 - 2x\right) = x^3 + 2x$

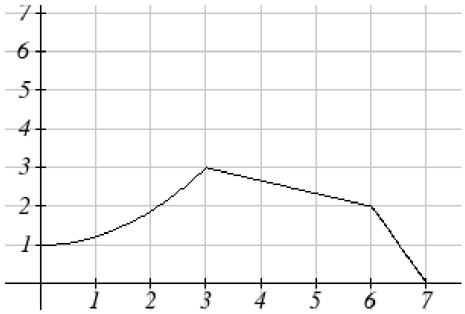
Since $-f(-x) = f(x)$, this is an odd function

Stretches and Compressions
With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the inputs or outputs.

Remember, we can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

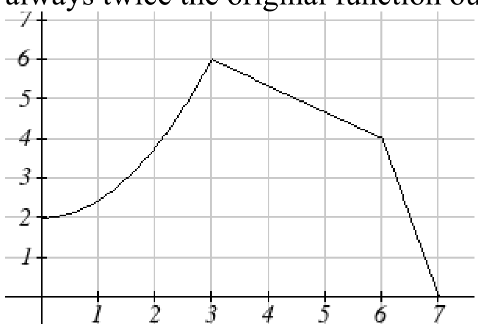
Example 13

A function *P(t)* models the growth of a population of fruit flies. The growth is shown below.



A scientist is comparing this to another population, *Q*, that grows the same way, but starts twice as large. Sketch a graph of this population.

Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like



Symbolically,
$Q(t) = 2P(t)$

This means that for any input *t*, the value of the *Q* function is twice the value of the *P* function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, *t*, stay the same while the output values are twice as large as before.

Vertical Stretch/Compression

Given a function *f(x)*, and if we define a new function *g(x)* as
$g(x) = kf(x)$, where *k* is a constant
then *g(x)* is a **vertical stretch or compression** of the function *f(x)*

If $k > 1$, then the graph will be stretched
If $0 < k < 1$, then the graph will be compressed
If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

Example 14

A function *f(x)* is given as a table below. Create a table for the function $g(x) = \frac{1}{2} f(x)$

x	2	4	6	8
$f(x)$	1	3	7	11

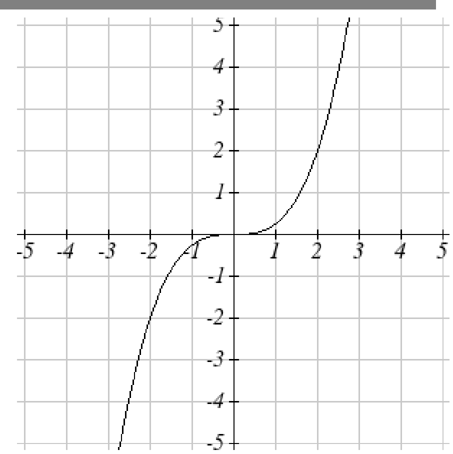
The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of *g* are half of the output value of *f* with the same input. For example, we know that $f(4) = 3$. Then
$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$

x	2	4	6	8
$g(x)$	1/2	3/2	7/2	11/2

The result is that the function *g(x)* has been compressed vertically by ½. Each output value has been cut in half, so the graph would now be half the original height.

Example 15

The graph to the right is a transformation of the toolkit function $f(x) = x^3$. Relate this new function *g(x)* to *f(x)*, then find a formula for *g(x)*.



When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of *g* are ¼ the outputs of the function *f*, since $g(2) = \frac{1}{4} f(2)$. From this we can fairly safely conclude that:
$g(x) = \frac{1}{4} f(x)$

We can write a formula for *g* by using the definition of the function *f*
$g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3$

Now we consider changes to the inside of a function

Example 16

Returning to the fruit fly population we looked at earlier, suppose the scientist is now comparing it to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, *R*, will progress in 1 hour the same amount the original population did in 2 hours, and in 2 hours, will progress as much as the original population did in 4 hours. Sketch a graph of this population.

Symbolically, we could write

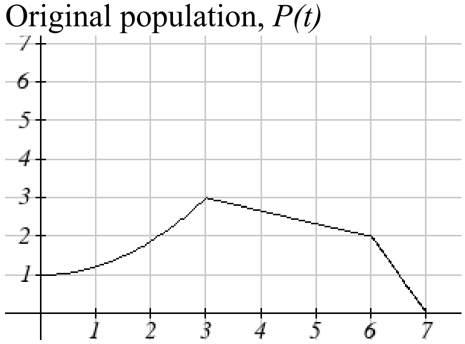
$R(1) = P(2)$

$R(2) = P(4)$, and in general,

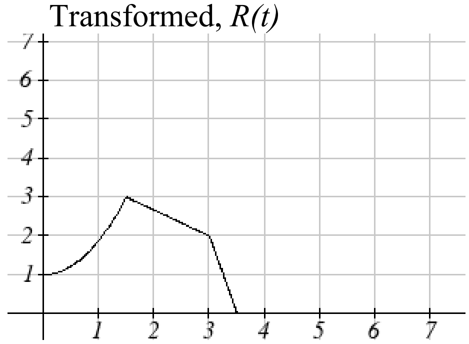
$R(t) = P(2t)$

Graphing this,

Original population, *P(t)*



Transformed, *R(t)*



Note the effect on the graph is a horizontal compression, where all input values are half their original distance from the vertical axis.

Horizontal Stretch/Compression

Given a function *f(x)*, and if we define a new function *g(x)* as

$g(x) = f(kx)$, where *k* is a constant

then *g(x)* is a **horizontal stretch or compression** of the function *f(x)*

If $k > 1$, then the graph will be compressed by $1/k$

If $0 < k < 1$, then the graph will be stretched by $1/k$

If $k < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example 17

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$

x	2	4	6	8
f(x)	1	3	7	11

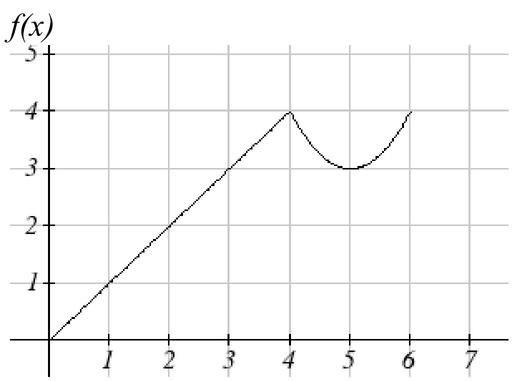
The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for *g* are the same as the output values for the function *f* at an input half the size. Notice that we don't have enough information to determine $g(2)$ since $g(2) = f\left(\frac{1}{2}\cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to *g* will need to be twice as large to get inputs for *f* that we can evaluate. For example, we can determine $g(4)$ since $g(4) = f\left(\frac{1}{2}\cdot 4\right) = f(2) = 1$.

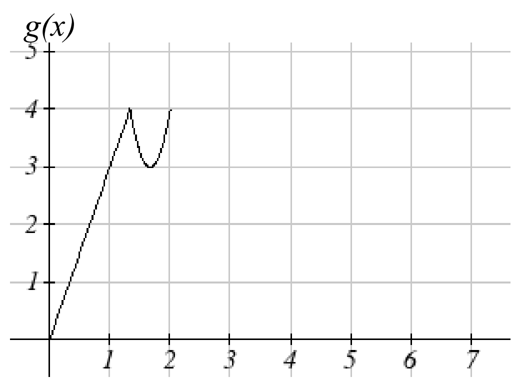
x	4	8	12	16
g(x)	1	3	7	11

Since each input value has been doubled, the result is that the function *g(x)* has been stretched horizontally by 2.

Example 18

Two graphs are shown below. Relate the function *g(x)* to *f(x)*





The graph of *g(x)* looks like the graph of *f(x)* horizontally compressed. Since *f(x)* ends at (6,4) and *g(x)* ends at (2,4) we can see that the *x* values have been compressed by 1/3, because 6(1/3) = 2. We might also notice that $g(2) = f(6)$, and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by 1/3.

Remember the coefficient needed for a horizontal stretch or compression is the *reciprocal* of the stretch or compression. So to stretch the graph horizontally by 4, we need a coefficient of 1/4 in our function: $f\left(\frac{1}{4}x\right)$. This means the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Try it Now

4. Write a formula for the toolkit square root function horizontally stretched by three.

It is good to note that for most toolkit functions, a horizontal stretch or vertical stretch can be represented in other ways. For example, a horizontal stretch of a power function can also be represented as a vertical stretch. When writing a formula for a transformed toolkit, we only need to find one transformation that would produce the graph.

Combining Transformations

When combining transformations, it is very important to consider order of transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3.

When we see an expression like $2f(x)+3$, which transformation should we start with?

The answer here follows nicely from order of operations, for outside transformations. Given the output value of *f(x)*, we first multiply by 2, causing the vertical stretch, then add 3, causing the vertical shift. (Multiplication before Addition)

Combining Vertical Transformations

When combining vertical transformations written in the form $af(x)+k$
First vertically stretch by *a*, then vertically shift by *k*

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$ for example, we have to think about how the inputs to the *g* function relate to the inputs to the *f* function. Suppose we know $f(7)=12$. What input to *g* would produce that output? In other words, what value of *x* will allow $g(x)=f(2x+3)=f(12)$? We would need $2x+3=12$. To solve for *x*, we would first subtract 3, resulting in horizontal shift, then divide by 2, causing a horizontal compression.

Combining Horizontal Transformations

When **combining horizontal transformations** written in the form $f(bx+p)$
First horizontally shift by p, then horizontally stretch by $1/b$

This format ends up being very difficult to work with, since it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right)$$

Factoring in this way allows us to horizontally stretch first then shift horizontally.

Combining Horizontal Transformations

When **combining horizontal transformations** written in the form $f(b(x+h))$
First horizontally stretch by $1/b$, then horizontally shift by h.

Independence of Horizontal and Vertical Transformations

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are done first.

Example 19

Given the table of values for the function *f(x)* below, create a table of values for the function $g(x)=2f(3x)+1$

x	6	12	18	24
$f(x)$	10	14	15	17

There are 3 steps to this transformation and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by 1/3 which means we multiply each x value by 1/3.

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We build this onto the previous transformation.

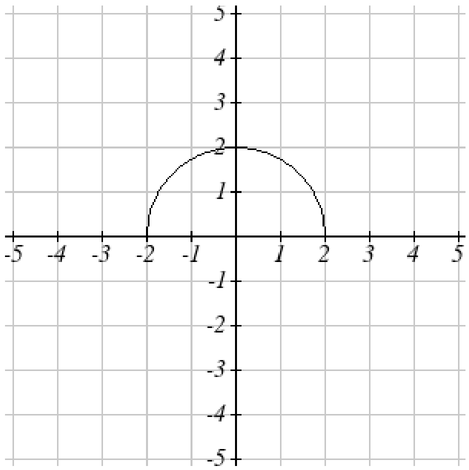
x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values.

x	2	4	6	8
$g(x)=2f(3x)+1$	21	29	31	35

Example 20

Using the graph of *f(x)* below, sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$



To make things simpler, we'll start by factoring out the inside of the function

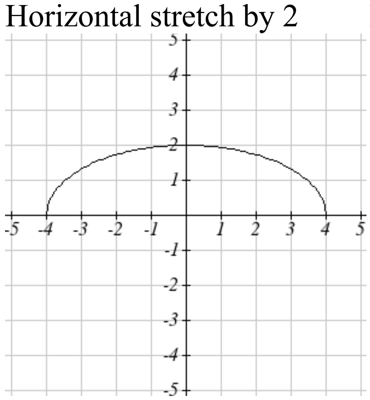
$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x+2)\right) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the ½ on the inside of the function. Remember twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0).

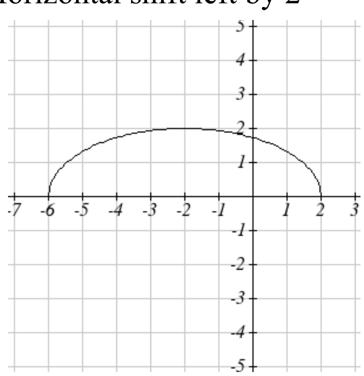
Next, we horizontally shift left by 2 units, as indicated by the *x*+2.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function.

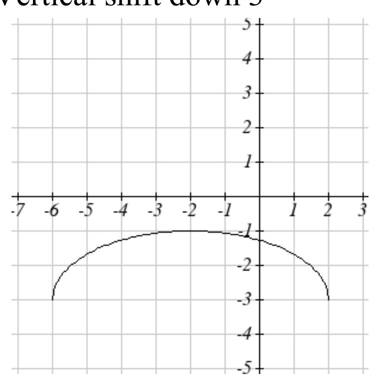
Horizontal stretch by 2



Horizontal shift left by 2

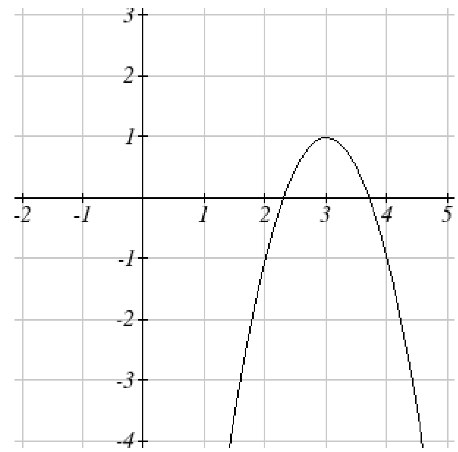


Vertical shift down 3



Example 21

Write an equation for the transformed graph of the quadratic function graphed to the right.



Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations:
The original tool kit function has been flipped over the *x* axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:
Vertical reflection, requiring a negative sign outside the function
Vertical Stretch *or* Horizontal Compression*
Horizontal Shift Right 3 units, which tells us to put *x*-3 on the inside of the function
Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

* It is unclear from the graph whether it is showing a vertical stretch or a horizontal compression. For the quadratic, it turns out we could represent it either way, so we'll use a vertical stretch. You may be able to determine the vertical stretch by observation.

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

Not everyone can see this by simply looking at the graph. If you can, great, but if not, we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch.

$f(x) = x^2$	The original function
$-f(x) = -x^2$	Vertically reflected
$-af(x) = -ax^2$	Vertically stretched
$-af(x-3) = -a(x-3)^2$	Shifted right 3
$-af(x-3)+1 = -a(x-3)^2 + 1$	Shifted up 1

We now know our graph is going to have an equation of the form $g(x) = -a(x-3)^2 + 1$.
To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2) = -1$. Using our general formula, and substituting 2 for *x*, and -1 for *g(x)*,

$-1 = -a(2-3)^2 + 1$

$-1 = -a + 1$

$-2 = -a$

$2 = a$

To produce the graph, we must have vertically stretched by two. Our final equation for this graph then is $g(x) = -2(x-3)^2 + 1$

Try it Now

5. Consider the linear function $g(x) = -2x + 1$. Describe its transformation in words using the identity tool kit function *f(x)* = *x* as a reference point.

Example 22

On what interval(s) is the function $g(x) = \dfrac{-2}{(x-1)^2} + 3$ increasing and decreasing?

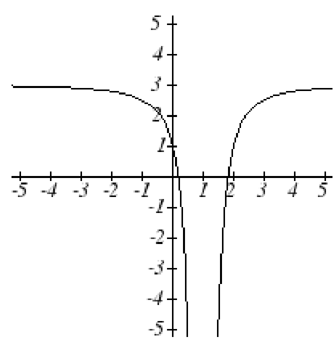
This is a transformation of the toolkit reciprocal squared function, $f(x) = \dfrac{1}{x^2}$:

$-2f(x) = \dfrac{-2}{x^2}$ A vertical flip and vertical stretch by 2

$-2f(x-1) = \dfrac{-2}{(x-1)^2}$ A shift right by 1

$-2f(x-1) + 3 = \dfrac{-2}{(x-1)^2} + 3$ A shift up by 3

The basic reciprocal squared function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Because of the vertical flip, the *g(x)* function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine *g(x)* will be increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. We also could graph the transformation to help us determine these intervals.



Try it Now

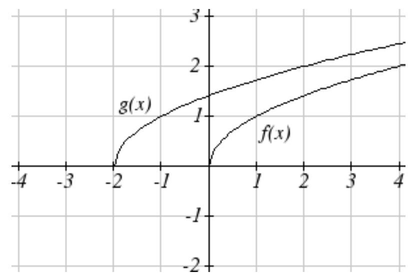
6. On what interval(s) is the function $h(t) = (t-3)^3 + 2$ concave up and down?

Important Topics of this Section

Transformations
Vertical Shift (up & down)
Horizontal Shifts (left & right)
Reflections over the vertical & horizontal axis
Even & Odd functions
Vertical Stretches & Compressions
Horizontal Stretches & Compressions
Combinations of Transformation

Try it Now Answers

1. $b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$
2. a. Horizontal shift
   b. The function is shifted to the LEFT by 2 units.
   c. Shown to the right



3. Graph of $f(x) = x^2$, and $g(x) = -f(x)$ and $h(x) = f(-x)$
   Notice: $f(-x)$ looks the same as $f(x)$
4. $g(x) = f\left(\frac{1}{3}x\right)$ so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$
5. The identity tool kit function $f(x) = x$ has been transformed in 3 steps
   a. Vertically stretched by 2.
   b. Vertically reflected over the x axis.
   c. Vertically shifted up by 1 unit.

6. $h(t)$ is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$

Section 1.5 Exercises

Describe how each function is a transformation of the original function $f(x)$

1. $f(x-49)$
2. $f(x+43)$
3. $f(x+3)$
4. $f(x-4)$
5. $f(x)+5$
6. $f(x)+8$
7. $f(x)-2$
8. $f(x)-7$
9. $f(x-2)+3$
10. $f(x+4)-1$

11. Write a formula for $f(x)=\sqrt{x}$ shifted up 1 unit and left 2 units
12. Write a formula for $f(x)=|x|$ shifted down 3 units and right 1 unit
13. Write a formula for $f(x)=\dfrac{1}{x}$ shifted down 4 units and right 3 units
14. Write a formula for $f(x)=\dfrac{1}{x^2}$ shifted up 2 units and left 4 units

15. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
f(x)	-2	-1	-3	1	2

x	-1	0	1	2	3
g(x)	-2	-1	-3	1	2

x	-2	-1	0	1	2
h(x)	-1	0	-2	2	3

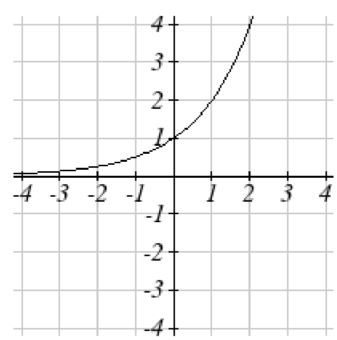
16. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
f(x)	-1	-3	4	2	1

x	-3	-2	-1	0	1
g(x)	-1	-3	4	2	1

x	-2	-1	0	1	2
h(x)	-2	-4	3	1	0

The graph of $f(x)=2^x$ is shown. Sketch a graph of each transformation of $f(x)$

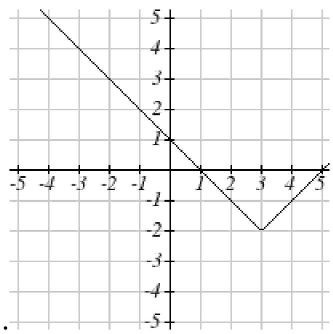


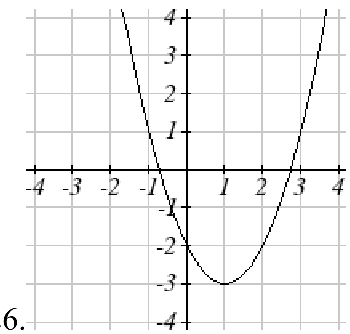
17. $g(x)=2^x+1$
18. $h(x)=2^x-3$
19. $w(x)=2^{x-1}$
20. $q(x)=2^{x+3}$

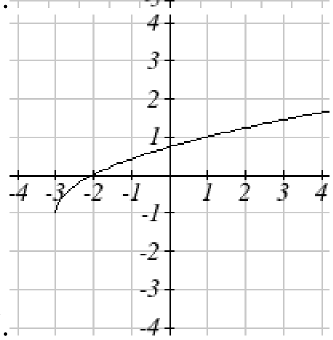
Sketch a graph of each function as a transformation of a toolkit function

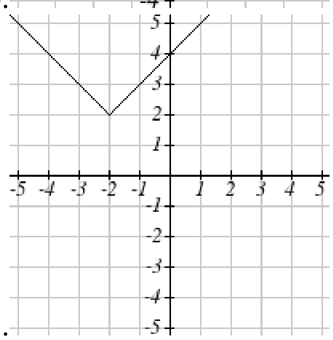
21. $f(t) = (t+1)^2 - 3$
22. $h(x) = |x-1| + 4$
23. $k(x) = (x-2)^3 - 1$
24. $m(t) = 3 + \sqrt{t+2}$

Write an equation for the function graphed below

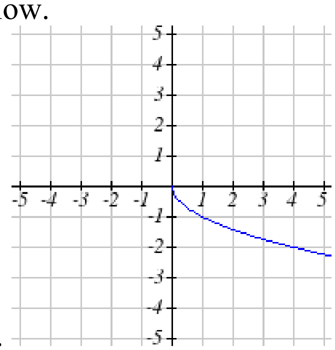
25. 

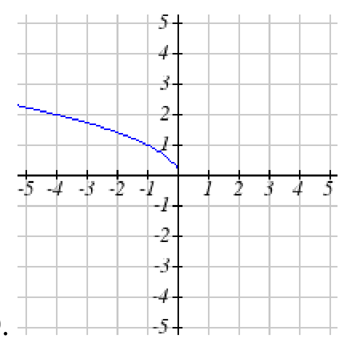
26. 

27. 

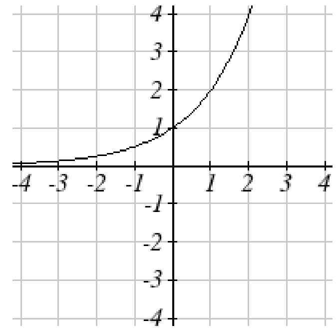
28. 

Find a formula for each of the transformations of the square root whose graphs are given below.

29. 

30. 

The graph of $f(x) = 2^x$ is shown. Sketch a graph of each transformation of $f(x)$

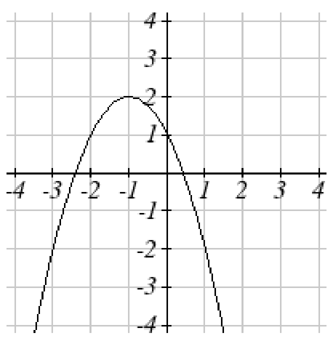


31. $g(x) = -2^x + 1$
32. $h(x) = 2^{-x}$

33. Starting with the graph of $f(x) = 6^x$ write the equation of the graph that results from
    a. reflecting $f(x)$ about the *x*-axis and the *y*-axis
    b. reflecting $f(x)$ about the *x*-axis, shifting left 2 units, and down 3 units

34. Starting with the graph of $f(x) = 4^x$ write the equation of the graph that results from
    a. reflecting $f(x)$ about the *x*-axis
    b. reflecting $f(x)$ about the *y*-axis, shifting right 4 units, and up 2 units

Write an equation for the function graphed below

35.


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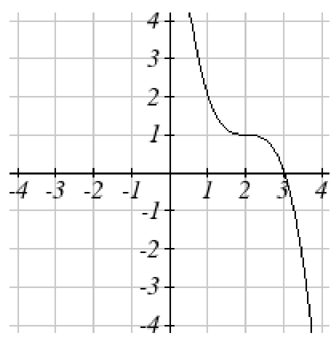

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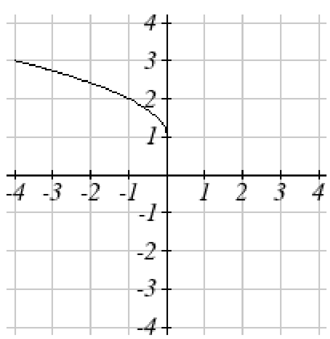

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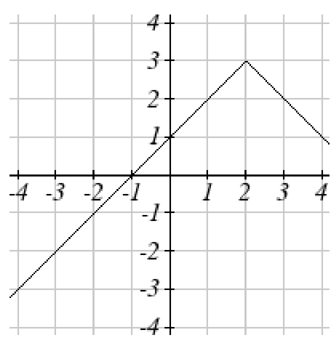


39. For each equation below, determine if the function is Odd, Even, or Neither

a. $f(x) = 3x^4$

b. $g(x) = \sqrt{x}$

c. $h(x) = \frac{1}{x} + 3x$

40. For each equation below, determine if the function is Odd, Even, or Neither

a. $f(x) = (x-2)^2$

b. $g(x) = 2x^4$

c. $h(x) = 2x - x^3$

Describe how each function is a transformation of the original function $f(x)$

41. $-f(x)$
42. $f(-x)$
43. $4f(x)$
44. $6f(x)$
45. $f(5x)$
46. $f(2x)$
47. $f\left(\frac{1}{3}x\right)$
48. $f\left(\frac{1}{5}x\right)$
49. $3f(-x)$
50. $-f(3x)$

51. Write a formula for $f(x) = |x|$ reflected over the *y* axis and horizontally compressed by a factor of $\frac{1}{4}$

52. Write a formula for $f(x) = \sqrt{x}$ reflected over the *x* axis and horizontally stretched by a factor of 2

53. Write a formula for $f(x) = \frac{1}{x^2}$ vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

54. Write a formula for $f(x) = \frac{1}{x}$ vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

55. Write a formula for $f(x) = x^2$ horizontally compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

56. Write a formula for $f(x) = x^2$ horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

Describe how each formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

57. $f(x)=4(x+1)^2-5$

58. $g(x)=5(x+3)^2-2$

59. $h(x)=-2|x-4|+3$

60. $k(x)=-3\sqrt{x}-1$

61. $m(x)=\frac{1}{2}x^3$

62. $n(x)=\frac{1}{3}|x-2|$

63. $p(x)=\left(\frac{1}{3}x\right)^2-3$

64. $q(x)=\left(\frac{1}{4}x\right)^3+1$

65. $a(x)=\sqrt{-x+4}$

66. $b(x)=\sqrt[3]{-x-6}$

Determine the interval(s) on which the function is increasing and decreasing

67. $f(x)=4(x+1)^2-5$

68. $g(x)=5(x+3)^2-2$

69. $a(x)=\sqrt{-x+4}$

70. $k(x)=-3\sqrt{x}-1$

Determine the interval(s) on which the function is concave up and concave down

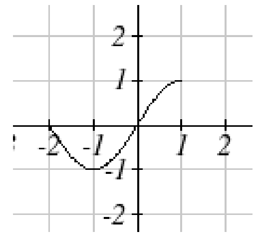
71. $m(x)=-2(x+3)^3+1$

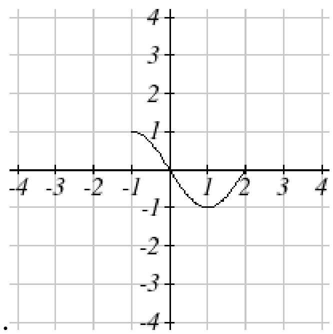
72. $b(x)=\sqrt[3]{-x-6}$

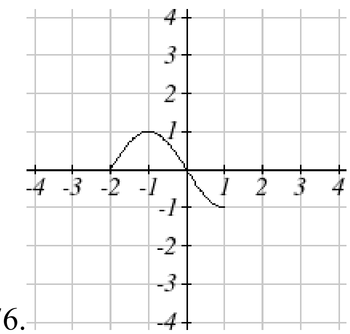
73. $p(x)=\left(\frac{1}{3}x\right)^2-3$

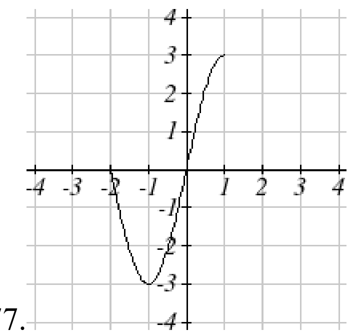
74. $k(x)=-3\sqrt{x}-1$

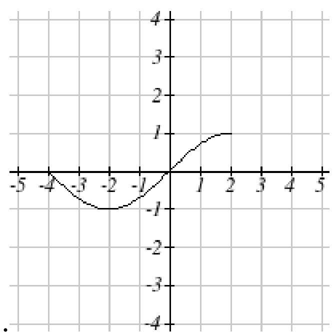
The function $f(x)$ is graphed here. Write an equation for each graph below as a transformation of $f(x)$.

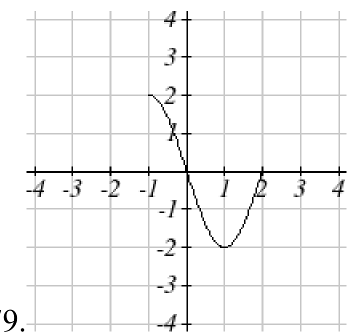


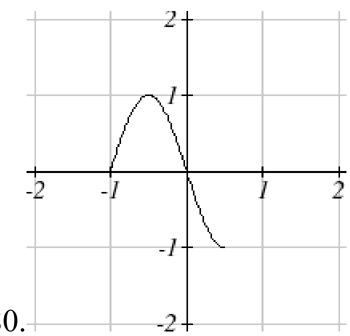
75. 

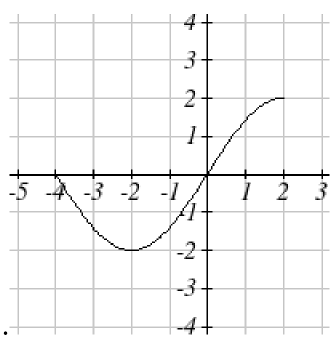
76. 

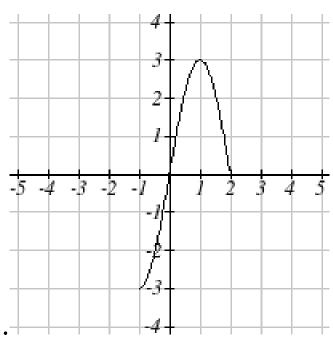
77. 

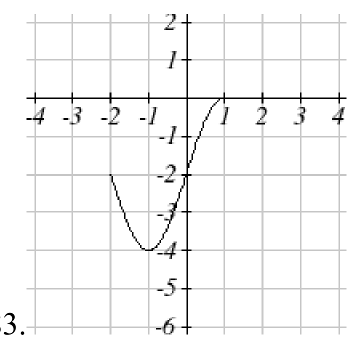
78. 

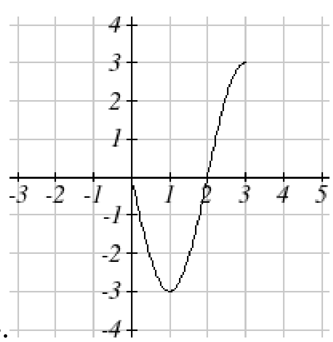
79. 

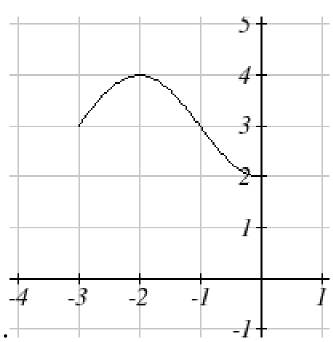
80. 

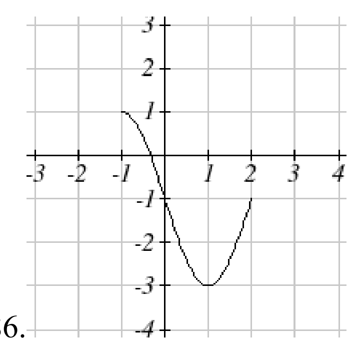
81. 

82. 

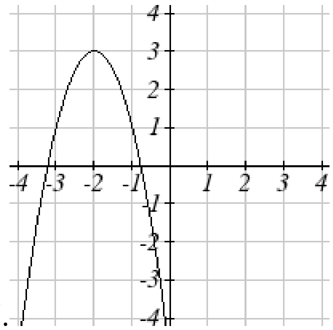
83. 

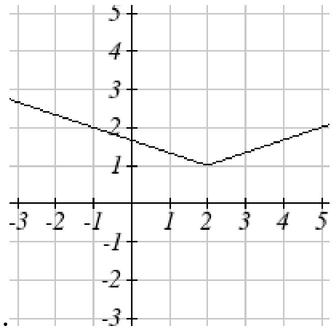
84. 

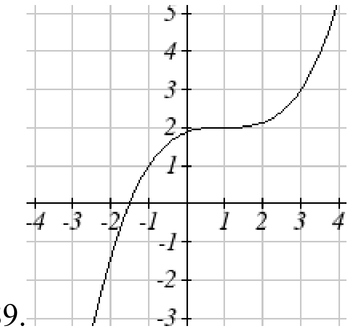
85. 

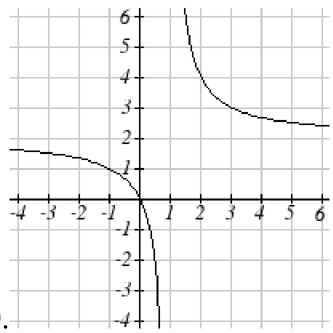
86. 

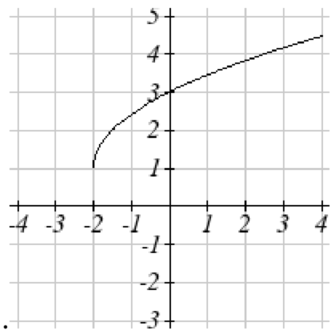
Write an equation for the transformed toolkit function graphed below.

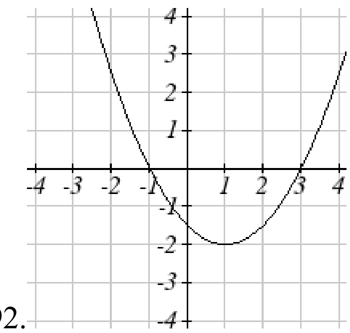
87. 

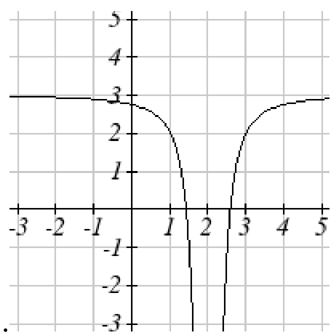
88. 

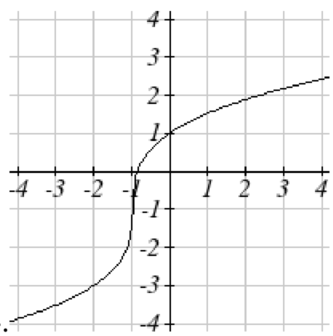
89. 

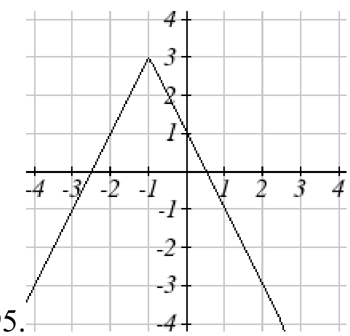
90. 

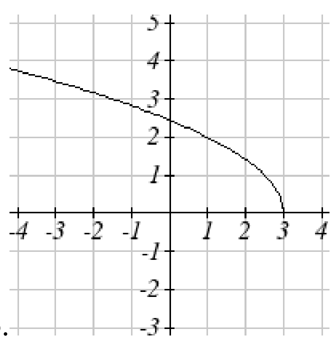
91. 

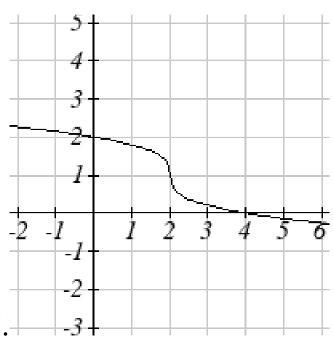
92. 

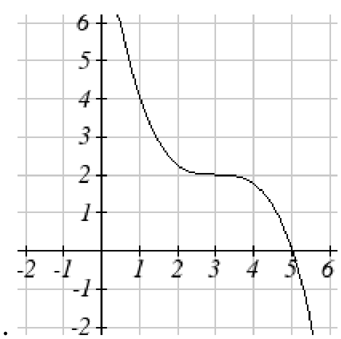
93. 

94. 

95. 

96. 

97. 

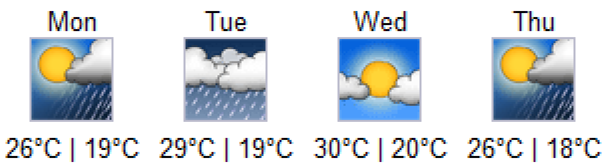
98. 

99. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \le x \le 6$ and the range of $f(x)$ is $-3 \le y \le 5$. [UW]

a. What is the domain of $f(2(x-3))$?
b. What is the range of $f(2(x-3))$?
c. What is the domain of $2f(x)-3$?
d. What is the range of $2f(x)-3$?
e. Can you find constants *B* and *C* so that the domain of $f(B(x-C))$ is $8 \le x \le 9$?
f. Can you find constants *A* and *D* so that the range of $Af(x)+D$ is $0 \le y \le 1$?

Section 1.6 Inverse Functions

A fashion designer is travelling to Milan for a fashion show. He asks his assistant, Betty, what 75 degrees Fahrenheit is in Celsius, and after a quick search on Google, she finds the formula $C = \frac{5}{9}(F - 32)$. Using this formula, she calculates $\frac{5}{9}(75 - 32) \approx 24$ degrees Celsius. The next day, the designer sends his assistant the week's weather forecast for Milan, and asks her to convert the temperatures to Fahrenheit.



At first, Betty might consider using the formula she has already found to do the conversions. After all, she knows her algebra well, and can easily solve the equation for *F* after substituting a value for *C*. For example, to convert 26 degrees Celsius, she could:

$$26 = \frac{5}{9}(F - 32)$$

$$26 \cdot \frac{9}{5} = F - 32$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, she realizes that solving the equation for each of the temperatures would get awfully tedious, and realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one which takes the Celsius temperature and outputs the Fahrenheit temperature. This is the idea of an inverse function, where input becomes output and the output becomes the input.

Inverse Function

If $f(a) = b$, then a function *g(x)* is an **inverse** of *f* if $g(b) = a$

The inverse of *f(x)* is typically notated $f^{-1}(x)$, which is read "*f* inverse of *x*", so equivalently, if $f(a) = b$ then $f^{-1}(b) = a$.

Important: The raised -1 used in the notation for inverse functions is simply a notation, and does not designate an exponent or power of -1.

Example 1

If for a particular function, $f(2) = 4$, what do we know about the inverse?

The inverse function reverses which quantity is input and which quantity is output, so if $f(2) = 4$, then $f^{-1}(4) = 2$.

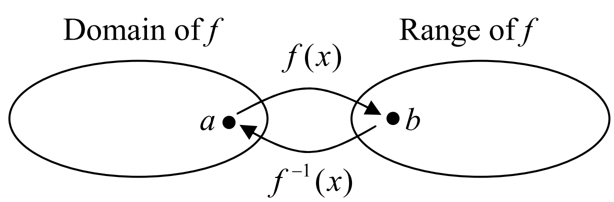
Alternatively, if you want to re-name the inverse function *g(x)*, then $g(4) = 2$

Try it Now

1. Given the inverse function $h^{-1}(6) = 2$, what do we know about the original function?

Notice that original function and the inverse function *undo* each other. If $f(a) = b$, then $f^{-1}(b) = a$, returning us to the original input. More simply put, if you compose these functions together you get the original input as your answer.

$f^{-1}\left(f(a)\right) = a$ and $f\left(f^{-1}(b)\right) = b$



Since the outputs of the function *f* are the inputs to f^{-1}, typically the range of *f* is also the domain of f^{-1}. Likewise, since the inputs to *f* are the outputs of f^{-1}, the domain of *f* is typically the range of f^{-1}.

Basically, like how the input and output values switch, the domain & ranges switch as well. But be careful, because sometimes a function doesn't even have an inverse function, or only has an inverse on a limited domain.

Example 2

The function $f(x) = 2^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$, what would we expect the domain and range of f^{-1} to be?

We would expect f^{-1} to swap the domain and range of *f*, so f^{-1} would have domain $(0, \infty)$ and range $(-\infty, \infty)$.

Example 3

A function *f(t)* is given as a table below, showing distance in miles that a car has traveled in *t* minutes. Find and interpret $f^{-1}(70)$

t (minutes)	30	50	70	90
f(t) (miles)	20	40	60	70

The inverse function takes an output of *f* and returns an input for *f*. So in the expression $f^{-1}(70)$, the 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function *f*, 90 minutes, so $f^{-1}(70) = 90$. Interpreting this, it means that to drive 70 miles, it took 90 minutes.

Alternatively, recall the definition of the inverse was that if $f(a) = b$ then $f^{-1}(b) = a$. By this definition, if you are given $f^{-1}(70) = a$ then you are looking for a value *a* so that $f(a) = 70$. In this case, we are looking for a *t* so that $f(t) = 70$, which is when *t* = 90.

Try it Now

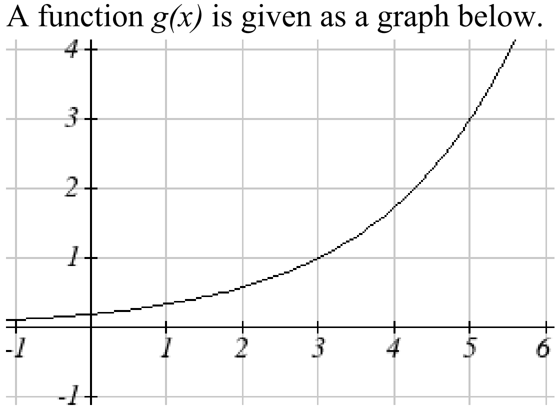
2. Using the table below

t (minutes)	30	50	60	70	90
f(t) (miles)	20	40	50	60	70

Find the following
   a. $f(60)$
   b. $f^{-1}(60)$

Example 4

A function *g(x)* is given as a graph below. Find $g(3)$ and $g^{-1}(3)$



To evaluate $g(3)$, we find 3 on the horizontal input axis and find the corresponding output value on the vertical output axis. The point (3, 1) tells us that $g(3) = 1$

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means g(*x*) = 3. By looking for the output value 3 on the vertical axis we find the point (5, 3) on the graph, which means g(5) = 3, so by definition $g^{-1}(3) = 5$.

Try it Now

3. Using the graph in example 4 above
   a. find $g^{-1}(1)$
   b. estimate $g^{-1}(4)$

Example 5

Returning to our designer's assistant, find a formula for the inverse function that gives Fahrenheit temperature given a Celsius temperature.

A quick Google search would find the inverse function, but alternatively, Betty might look back at how she solved for the Fahrenheit temperature for a specific Celsius value, and repeat the process in general

$$C = \frac{5}{9}(F - 32)$$

$$C \cdot \frac{9}{5} = F - 32$$

$$F = \frac{9}{5}C + 32$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32)$$

Then

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function *h* to represent the conversion since the input and output variables are descriptive, and writing C^{-1} could get confusing.

It is important to note that not all functions will have an inverse function. Since the inverse $f^{-1}(x)$ takes an output of *f* and returns an input of *f*, in order for f^{-1} to itself be a function, then each output of *f* (input to f^{-1}) must correspond to exactly one input of *f* (output of f^{-1}) in order for f^{-1} to be a function. You might recall that this is the definition of a one-to-one function.

Properties of Inverses

In order **for a function to have an inverse**, it must be a one-to-one function

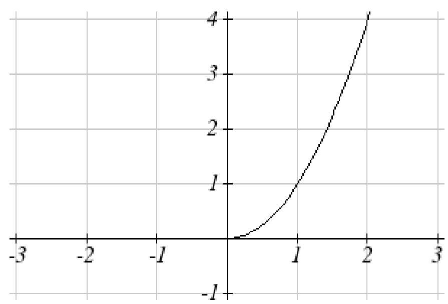
In some cases, it is desirable to have an inverse for a function even though the function is not one-to-one. In those cases, we can often limit the domain of the original function to an interval on which the function *is* one-to-one, then find an inverse only on that interval.

If you have not already done so, go back to the toolkit functions that were not one-to-one and limit or restrict the domain of the original function so that it is one-to-one. If you are not sure how to do this, proceed to example 6.

Example 6

The quadratic function $h(x) = x^2$ is not one-to-one. Find a domain on which this function is one-to-one, and find the inverse on that domain.

We can limit the domain to $[0, \infty)$ to restrict the graph to a portion that is one-to-one, and find its inverse on this limited domain.



You may have already guessed that since we undo a square with a square root, the inverse of $h(x) = x^2$ on this domain is $h^{-1}(x) = \sqrt{x}$.

You can also solve for the inverse function algebraically. If $h(x) = x^2$, we can introduce the variable y to represent the output values, allowing us to write $y = x^2$. To find the inverse we solve for the input variable

To solve for x we take the square root of each side. $\sqrt{y} = \sqrt{x^2}$ and get $\sqrt{y} = \pm x$ but we are only interested in the positive half so $x = \sqrt{y}$ or $h^{-1}(y) = \sqrt{y}$. In cases like this where the variables are not descriptive, it is common to see the inverse function rewritten with the variable x: $h^{-1}(x) = \sqrt{x}$. Rewriting the inverse using the variable x is often required for graphing inverse functions using calculators or computers.

Note that the domain and range of the square root function do correspond with the range and domain of the quadratic function on the limited domain.

Important Topics of this Section

Definition of an inverse function
Composition of inverse functions yield the original input value
Not every function has an inverse function
To have an inverse a function must be one-to-one
Restricting the domain of functions that are not one-to-one.

Try it Now Answers

1. $g(2) = 6$

2.a. $f(60) = 50$

b. $f^{-1}(60) = 70$

3. a. $g^{-1}(1) = 3$

b. $g^{-1}(4) = 5.5$ (this is an approximation – answers may vary slightly)

Section 1.6 Exercises

Assume that the function *f* is a one-to-one function.

1. If $f(6)=7$, find $f^{-1}(7)$
2. If $f(3)=2$, find $f^{-1}(2)$
3. If $f^{-1}(-4)=-8$, find $f(-8)$
4. If $f^{-1}(-2)=-1$, find $f(-1)$
5. If $f(5)=2$, find $(f(5))^{-1}$
6. If $f(1)=4$, find $(f(1))^{-1}$

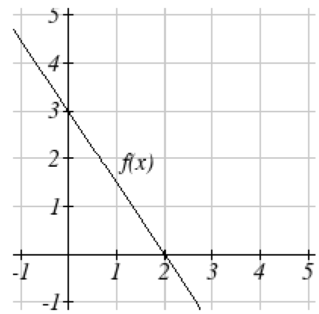
7. Using the graph of $f(x)$ shown

   a. Find $f(0)$

   b. Solve $f(x)=0$

   c. Find $f^{-1}(0)$

   d. Solve $f^{-1}(x)=0$



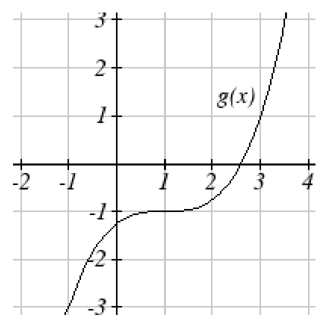
8. Using the graph shown

   a. Find $g(1)$

   b. Solve $g(x)=1$

   c. Find $g^{-1}(1)$

   d. Solve $g^{-1}(x)=1$



9. Use the table below to fill in the missing values.

x	0	1	2	3	4	5	6	7	8	9
f(x)	8	0	7	4	2	6	5	3	9	1

   a. Find $f(1)$

   b. Solve $f(x)=3$

   c. Find $f^{-1}(0)$

   d. Solve $f^{-1}(x)=7$

10. Use the table below to fill in the missing values.

t	0	1	2	3	4	5	6	7	8
h(t)	6	0	1	7	2	3	5	4	9

a. Find $h(6)$

b. Solve $h(t)=0$

c. Find $h^{-1}(5)$

d. Solve $h^{-1}(t)=1$

For each table below, create a table for $f^{-1}(x)$.

11.

x	3	6	9	13	14
f(x)	1	4	7	12	16

12.

x	3	5	7	13	15
f(x)	2	6	9	11	16

For each function below, find $f^{-1}(x)$

13. $f(x)=x+3$

14. $f(x)=x+5$

15. $f(x)=2-x$

16. $f(x)=3-x$

17. $f(x)=11x+7$

18. $f(x)=9+10x$

For each function, find a domain on which *f* is one-to-one and non-decreasing, then find the inverse of *f* restricted to that domain.

19. $f(x)=(x+7)^2$

20. $f(x)=(x-6)^2$

21. $f(x)=x^2-5$

22. $f(x)=x^2+1$

23. If $f(x)=x^3-5$ and $g(x)=\sqrt[3]{x+5}$, find

a. $f(g(x))$

b. $g(f(x))$

c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

24. If $f(x)=\dfrac{x}{2+x}$ and $g(x)=\dfrac{2x}{1-x}$, find

a. $f(g(x))$

b. $g(f(x))$

c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

Chapter 2: Linear Functions

Chapter one is a window that gives us a peek into the entire course. Our goal is to understand the basic structure of functions and function notation, the toolkit functions, domain and range, how to recognize and understand composition and transformations of functions and how to understand and utilize inverse functions. With these basic components in hand we will further research the specific details and intricacies of each type of function in our toolkit and use them to model the world around us.

Mathematical Modeling

As we approach day to day life we often need to quantify the things around us, giving structure and numeric value to various situations. This ability to add structure enables us to make choices based on patterns we see that are weighted and systematic. With this structure in place we can model and even predict behavior to make decisions. Adding a numerical structure to a real world situation is called **Mathematical Modeling.**

When modeling real world scenarios, there are some common growth patterns that are regularly observed. We will devote this chapter and the rest of the book to the study of the functions used to model these growth patterns.

Section 2.1 Linear Functions

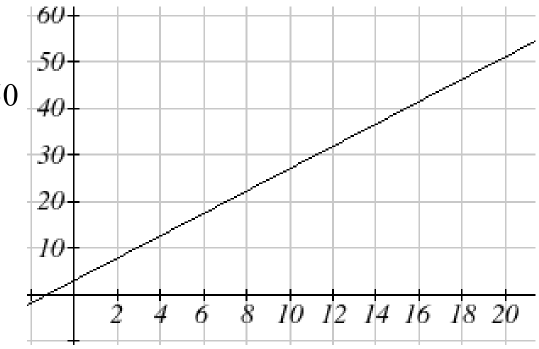
As you hop into a taxicab in Las Vegas, the meter will immediately read \$3.30, this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.40 for each mile the taxi drives[1]. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose *m* for miles and *C* for Cost in dollars as a function of miles: *C(m)*.

We know for certain that $C(0) = 3.30$, since the \$3.30 drop charge is assessed regardless of how many miles are driven. Since \$2.40 is added for each mile driven, then
$C(1) = 3.30 + 2.40 = 5.70$
If we then drove a second mile, another \$2.40 would be added to the cost:
$C(2) = 3.30 + 2.40 + 2.40 = 3.30 + 2.40(2) = 8.10$

---

[1] http://taxi.state.nv.us/FaresFees.htm, retrieved July 28, 2010. There is also a waiting fee assessed when the taxi is waiting at red lights, but we'll ignore that in this discussion.

If we drove a third mile, another \$2.40 would be added to the cost:
$C(3) = 3.30 + 2.40 + 2.40 + 2.40 = 3.30 + 2.40(3) = 10.50$



From this we might observe the pattern, and conclude that if m miles are driven, $C(m) = 3.30 + 2.40m$ because we start with a \$3.30 drop fee and then for each mile increase we add \$2.40.

It is good to verify that the units make sense in this equation. The \$3.30 drop charge is measured in dollars; the \$2.40 charge is measured in dollars per mile. So

$$C(m) = 3.30 dollars + \left(2.40 \frac{dollars}{mile}\right)(m\ miles)$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the *C* function.

Notice this equation $C(m) = 3.30 + 2.40m$ consisted of two quantities. The first is the fixed \$3.30 charge which does not change based on the value of the input. The second is the \$2.40 dollars per mile value, which is a **rate of change**. In the equation this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by \$2.40 for every 1 mile increase.

m	0	1	2	3
C(m)	3.30	5.70	8.10	10.50

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.30 + 2.40m$ we see the shape is a line, which is how these functions get their name: **linear functions**

When the number of miles is zero the cost is \$3.30, giving the point (0, 3.30) on the graph. This is the vertical or *C(m)* intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by \$2.40; this rate remains consistent.

In this example you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

Linear Function

A **linear function**. is a function whose graph produces a line. Linear functions can always be written in the form

$f(x) = b + mx$ or $f(x) = mx + b$; they're equivalent

Where

b is the initial or starting value of the function (when input, $x = 0$), and

m is the constant rate of change of the function

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

For this reason alone we will use the $f(x) = b + mx$ form for many of the examples, but remember they are equivalent and can be written correctly both ways.

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function.

$f(x) = b + mx$ is an **increasing** function if $m > 0$

$f(x) = b + mx$ is a **decreasing** function if $m < 0$

If $m = 0$, the rate of change zero, and the function $f(x) = b + 0x = b$ is just a straight horizontal line passing through the point (0, b), neither increasing nor decreasing.

Example 1

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?

The initial value for this function is 200, since he currently owns 200 songs so $N(0) = 200$. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula: $N(m) = 200 + 15m$.

N(m) is an increasing linear function.

With this formula we can predict how many songs he will have in 1 year (12 months): $N(12) = 200 + 15(12) = 200 + 180 = 380$. Marcus will have 380 songs in 12 months.

Try it Now

1. If you earn \$30,000 per year and you spend \$29,000 per year write an equation for the amount of money you save after y years, if you start with nothing.
"The most important thing, spend less than you earn![2]*"*

Calculating Rate of Change

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, $(x_1,\ y_1)$ and $(x_2,\ y_2)$, if we wish to find a linear function that contains both points we can calculate the rate of change, m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 2

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400\,people}{4\,years} = 1100\,\frac{people}{year} = 1100 \text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Example 3

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the components of this function.

[2] http://www.thesimpledollar.com/onepage

The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 4

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find the rate of change.

$f(3) = -2$ tells us that the input 3 corresponds with the output -2, and $f(8) = 1$ tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$. If desired we could also write this as $m = 0.6$

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

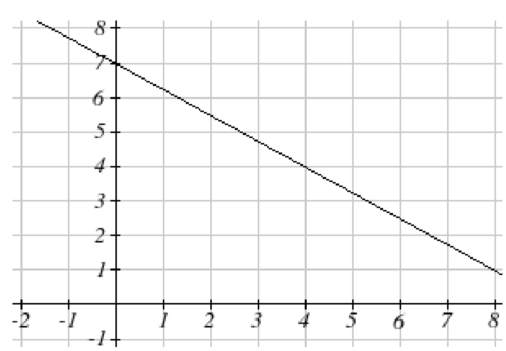
Try it Now

2. Given the two points (2, 3) and (0, 4), find the rate of change. Is this function increasing or decreasing?

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 5

Write an equation for the linear function graphed to the right.



Looking at the graph, we might notice that it passes through the points (0, 7) and (4, 4). From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$

Example 6

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

In example 3, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3.

$f(3) = b + \frac{3}{5}(3)$ Since we know that $f(3) = -2$, we can substitute on the left side

$-2 = b + \frac{3}{5}(3)$ This leaves us with an equation we can solve for the initial value

$b = -2 - \frac{9}{5} = \frac{-19}{5}$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = \frac{-19}{5} + \frac{3}{5}x$$

Example 7

Working as an insurance salesperson, Ilya earns a base salaray and a commission on each new polity, so Ilya's weekly income, *I*, depends on the number of new policies, *n*, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for *I(n)*, and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change.

$$m = \frac{920-760}{5-3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value

$I(n) = b + 80n$ then when $n = 3$, $I(3) = 760$, giving

$760 = b + 80(3)$ this allows us to solve for b

$b = 760 - 80(3) = 520$

This value is the starting value for the function. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Writing the final equation:

$I(n) = 520 + 80n$

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week.

Flashback

Looking at Example 7:
Determine the independent and dependent variables?
What is a reasonable domain and range?
Is this function one-to-one?

Try it Now

3. The balance in your college payment account C, is a function on the amount, a, you withdraw each quarter. Interpret the function $C(a) = 20000 - 4000a$ in words. How many quarters of college can you pay for until this account is empty?

Example 8

Given the table below write a linear equation that represents the table values

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

We can see from the table that the initial value of rats is 1000 so in the linear format $P(w) = b + mw$, $b = 1000$.

Rather than solving for m, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6.

The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write
$P(w) = b + mw$ as $P(w) = 1000 + 40w$

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),
$m = \frac{1240 - 1080}{6 - 2} = \frac{160}{4} = 40$ rats per week

Important Topics of this Section

Definition of Modeling
Definition of a linear function
Structure of a linear function
Increasing & Decreasing functions
Finding the vertical intercept (0, *b*)
Finding the slope/ rate of change, *m*
Interpreting linear functions

Try it Now Answers

1. $S(y) = 30{,}000y - 29{,}000y = 1000y$ $1000 is saved each year.
2. $m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; Decreasing because $m < 0$
3. Your College account starts with $20,000 in it and you withdraw $4,000 each quarter (or your account contains $20,000 and decreases by $4000 each quarter.) You can pay for 5 quarters before the money in this account is gone.

Flashback Answers

n (number of policies sold) is the independent variable
I(n) (weekly income as a function of policies sold) is the dependent variable.

A reasonable domain is (0, 15)*
A reasonable range is ($540, $1740)*
*answers may vary given reasoning is stated; 15 is an arbitrary upper limit based on selling 3 policies per day in a 5 day work week and $1740 corresponds with the domain.

Yes this function is one-to-one

Section 2.1 Exercises

1. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation $P(t)$, for the population t years after 2003.

2. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation $P(t)$, for the population t years after 2005.

3. Sonya is currently 10 miles from home, and is walking further away at 2 miles per hour. Write an equation for her distance from home t hours from now.

4. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

5. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

6. At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

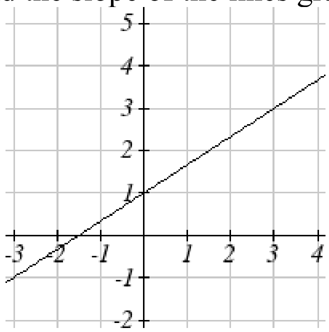
Determine if each function is increasing or decreasing

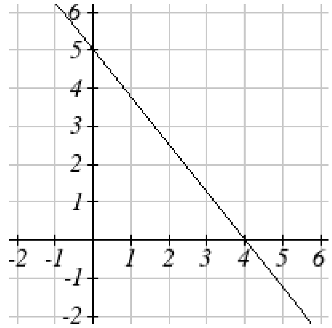
7. $f(x) = 4x + 3$
8. $g(x) = 5x + 6$
9. $a(x) = 5 - 2x$
10. $b(x) = 8 - 3x$
11. $h(x) = -2x + 4$
12. $k(x) = -4x + 1$
13. $j(x) = \frac{1}{2}x - 3$
14. $p(x) = \frac{1}{4}x - 5$
15. $n(x) = -\frac{1}{3}x - 2$
16. $m(x) = -\frac{3}{8}x + 3$

Find the slope of the line that passes through the two given points

17. (2, 4) and (4, 10)
18. (1, 5) and (4, 11)
19. (-1,4) and (5, 2)
20. (-2, 8) and (4, 6)
21. (6,11) and (-4,3)
22. (9,10) and (-6,-12)

Find the slope of the lines graphed

23. 

24. 

25. Sonya is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate?

26. A gym membership with two personal training sessions costs \$125, while gym membership with 5 personal training sessions costs \$260. What is the rate for personal training sessions?

27. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

28. A city's population in the year 1958 was 2,113,000. In 1991 the population was 2,099,800. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

29. A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

30. A phone company charges for service according to the formula: $C(n) = 26 + 0.04n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

31. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting point and how it is changing over time.

32. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after *t* minutes is given by $E(t)1200+40t$. Write a complete sentence describing Maria's starting point and how it is changing over time.

Given each set of information, find a linear equation satisfying the conditions, if possible

33. $f(-5)=-4$, and $f(5)=2$

34. $f(-1)=4$, and $f(5)=1$

35. Passes through (2, 4) and (4, 10)

36. Passes through (1, 5) and (4, 11)

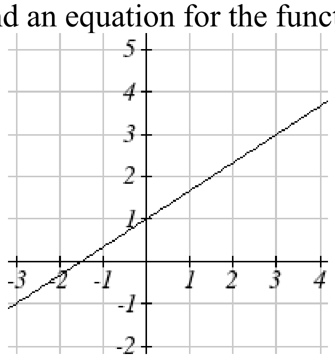
37. Passes through (-1,4) and (5, 2)

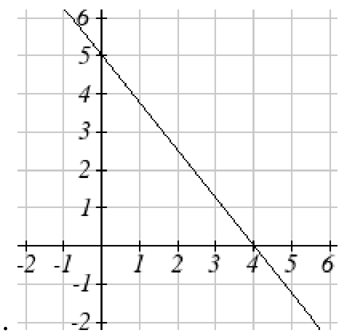
38. Passes through (-2, 8) and (4, 6)

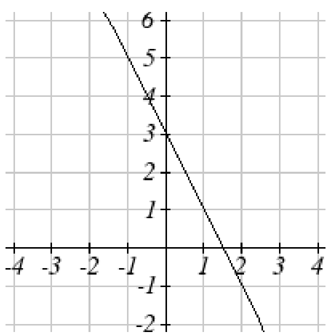
39. *x* intercept at (-2, 0) and *y* intercept at (0, -3)

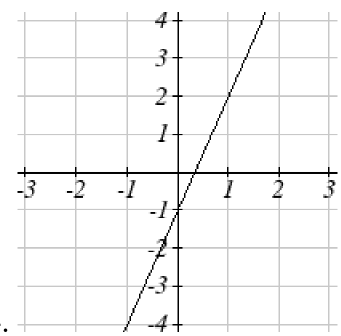
40. *x* intercept at (-5, 0) and *y* intercept at (0, 4)

Find an equation for the function graphed

41. 

42. 

43. 

44. 

45. A clothing business finds there is a linear relationship between the number of shirts, *n*, it can sell and the price, *p*, it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of \$30, while 3000 shirts can be sold at a price of \$22. Find a linear equation in the form $p = mn + b$ that gives the price *p* they can charge for *n* shirts.

46. A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

47. Which of the following tables which could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	5
5	-10
10	-25
15	-40

x	*h(x)*
0	5
5	30
10	105
15	230

x	*f(x)*
0	-5
5	20
10	45
15	70

x	*k(x)*
5	13
10	28
20	58
25	73

48. Which of the following tables which could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	6
2	-19
4	-44
6	-69

x	*h(x)*
2	13
4	23
8	43
10	53

x	*f(x)*
2	-4
4	16
6	36
8	56

x	*k(x)*
0	6
2	31
6	106
8	231

49. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was -23 Celsius (-23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees °F, but you could not find a reference with the correct formulas. You then remembered that the relationship between °F and °C is linear. [UW]
   a. Using this and the knowledge that 32°F = 0 °C and 212 °F = 100 °C, find an equation that computes Celsius temperature in terms of Fahrenheit; i.e. an equation of the form C = "an expression involving only the variable F."
   b. Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F = "an expression involving only the variable C."
   c. How cold was it in Oslo in °F?

Section 2.2 Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviors of the function. We will begin this exploration of linear functions with a look at graphs.

When graphing a linear function, there are three basic ways to graph it:

1) By plotting points (at least 2) and drawing a line through the points
2) Using the initial value and rate of change (slope)
3) Using transformations of the identity function $f(x) = x$

Example 1

Graph $f(x) = 5 - \frac{2}{3}x$ by plotting points

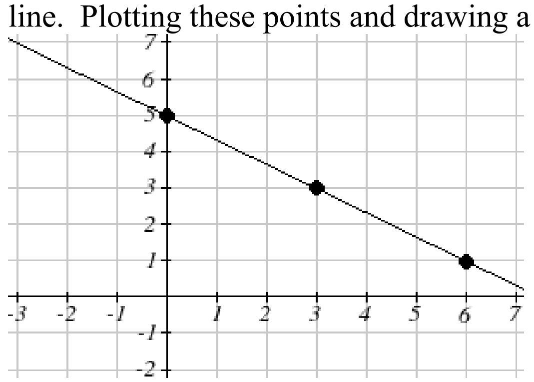
In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will "work nicely" in the equation. In this equation, multiples of 3 will work nicely due to the 2/3 in the equation, and of course using $x = 0$ to get the vertical intercept. Evaluating *f(x)* at $x = 0$, 3 and 6:

$$f(0) = 5 - \frac{2}{3}(0) = 5$$

$$f(3) = 5 - \frac{2}{3}(3) = 3$$

$$f(6) = 5 - \frac{2}{3}(6) = 1$$

These evaluations tell us that the points (0,5), (3,3), and (6,1) lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph



When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is zero, so in the equation $f(x) = b + mx$, the graph includes the point (0, *b*). On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

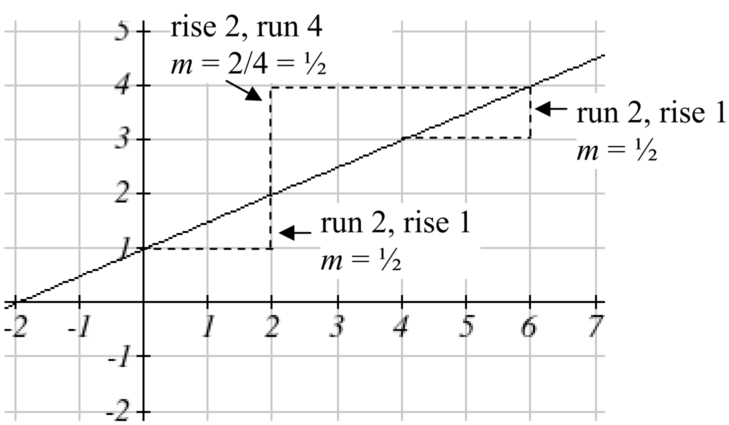
For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{\text{change of output}}{\text{change of input}}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{rise}{run}$$

Notice that this ratio is the same regardless of which two points we use



Graphical Interpretation of a Linear Equation

Graphically, in the equation $f(x) = b + mx$

b is the **vertical intercept** of the graph and tells us we can start our graph at (0, *b*)

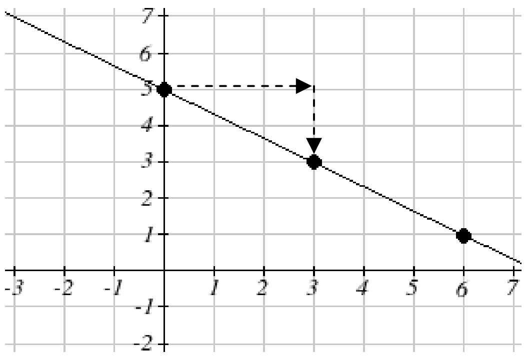
m is the **slope of the line** and tells us how far to rise & run to get to the next point

Example 2

Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.

The vertical intercept of the function is (0, 5), giving us a point on the graph of the line.

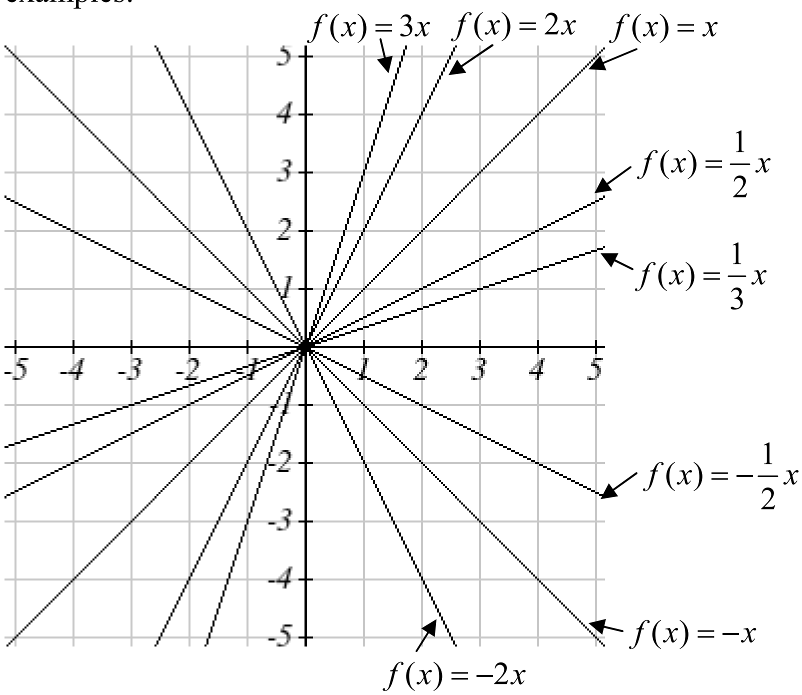
The slope is $-\frac{2}{3}$. This tells us that for every 3 units the graphs "runs" in the horizontal, the vertical "rise" decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value (0, 5) the slope tells us that if we move to the right 3, we will move down 2, moving us to the point (3, 3). We can continue this again to find a third point at (6, 1).



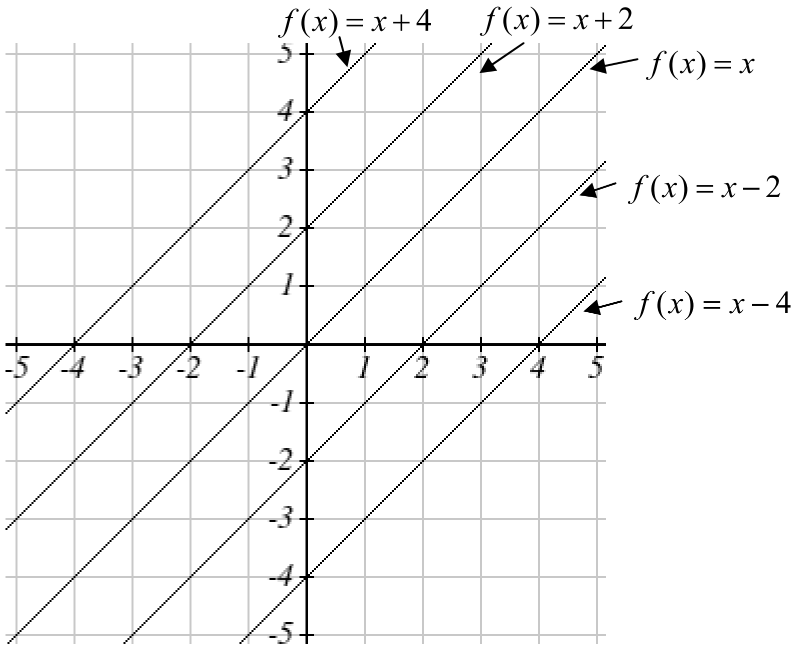
Try it Now

1. Consider that the slope -2/3 could also be written as 2/-3 . Using 2/-3, find another point on the graph that has a negative x value.

Another option for graphing is to use transformations of the identity function $f(x) = x$. In the equation $f(x) = mx$, the m is acting as the vertical stretch of the identity function. When m is negative, there is also a vertical reflection of the graph. Looking at some examples:



In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Some examples:



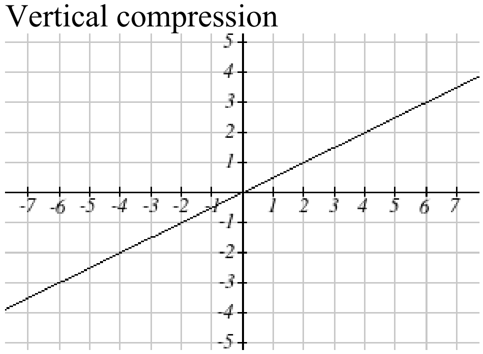
Using Vertical Stretches or Compressions along with Vertical Shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way for you to graph this type of function, make sure you practice each method.

Example 3

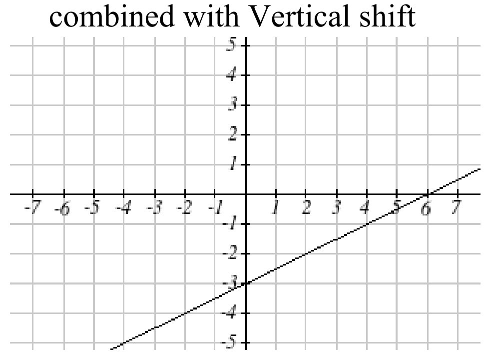
Graph $f(x) = -3 + \frac{1}{2}x$ using transformations.

The equation is the graph of the identity function vertically compressed by ½ and vertically shifted down 3.

Vertical compression



combined with Vertical shift



Notice how this nicely compares to the other method where the vertical intercept is found at (0, -3) and to get to the next point we rise (go up vertically) by 1 unit and run (go horizontally) by 2 units to get to the next point (2,-2), and the next one (4, -1). In these three points (0,-3), (2, -2), and (4, -1), the output values change by +1, and the *x* values change by +2, corresponding with the slope $m = 1/2$.

Example 4

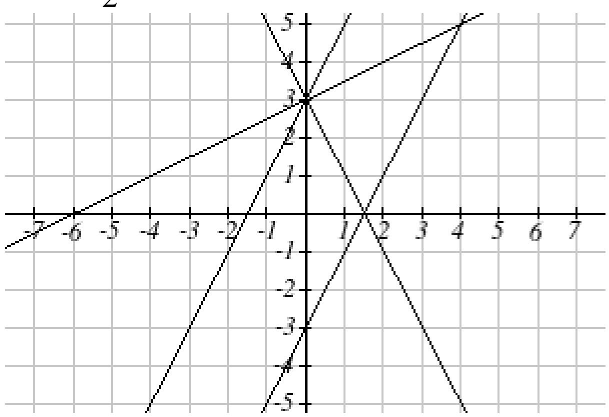
Match each equation with one of the lines in the graph below

$f(x) = 2x + 3$

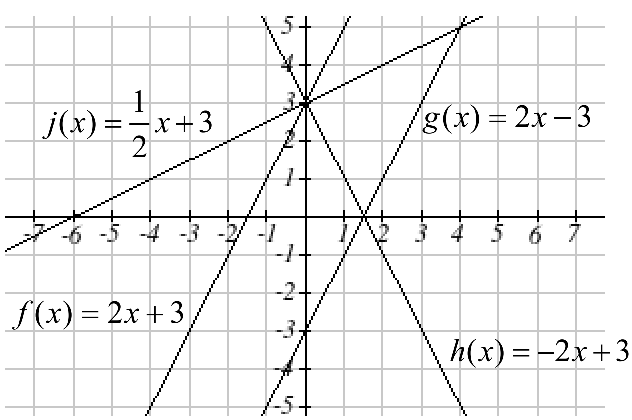
$g(x) = 2x - 3$

$h(x) = -2x + 3$

$j(x) = \frac{1}{2}x + 3$



Only one graph has a vertical intercept of -3, so we can immediately match that graph with *g(x)*. For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with *h(x)*. Of the other two, the steeper line would have a larger slope, so we can match that graph with equation *f(x)*, and the flatter line with the equation *j(x)*.



In addition to understanding the basic behavior of a linear function, increasing or decreasing and recognizing the slope and vertical intercept, it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Finding Horizontal Intercept

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. It can be found for any function by solving *f(x)* = 0.

Example 5

Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$

Setting the function equal to zero to find what input will put us on the horizontal axis,

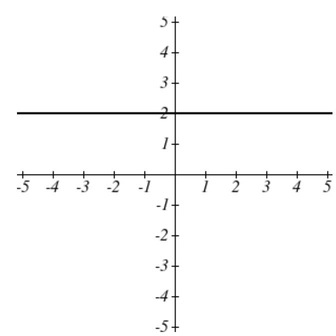
$$0 = -3 + \frac{1}{2}x$$

$$3 = \frac{1}{2}x$$

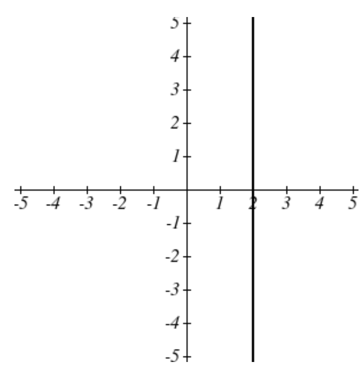
$$x = 6$$

The graph crosses the horizontal axis at (6,0)

There are two special cases of lines: a horizontal line and a vertical line. In a horizontal line like the one graphed to the right, notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0. Using a slope of 0 in the $f(x) = b + mx$, the equation simplifies to $f(x) = b$.



In the case of a vertical line, notice that between any two points, the change in the inputs is zero. In the slope equation, the denominator will be zero, and you may recall that we cannot divide by the zero; the slope of a vertical line is undefined. You might also notice that a vertical line is not a function. To write the equation of vertical line, we simply write input=value.



Horizontal and Vertical Lines

Horizontal lines have equations of the form $f(x) = b$

Vertical lines have equations of the form $x = a$

Example 6

Write an equation for the horizontal line graphed above.

This line would have equation $f(x) = 2$

Example 7

Write an equation for the vertical line graphed above.

This line would have equation $x = 2$

Try it Now

2. Describe the function $f(x) = 6 - 3x$ in terms of transformations of the identity function and find its horizontal intercept.

Parallel and Perpendicular Lines

When two lines are graphed at the same time, the lines will be **parallel** if they are increasing at the same rate – if the rates of change are the same. In this case, the graphs will never cross.

Parallel Lines

Two lines are **parallel** if the slopes are equal. In other words, given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$

The lines will be parallel if $m_1 = m_2$

Example 8

Find a line parallel to $f(x) = 6 + 3x$ that passes through the point (3, 0)

We know the line we're looking for will have the same slope as the given line, $m = 3$. Using this and the given point, we can solve for the new line's vertical intercept:

$g(x) = b + 3x$ then at (3, 0),

$0 = b + 3(3)$

$b = -9$

The line we're looking for is $g(x) = -9 + 3x$

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

Perpendicular Lines

Given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$

The lines will be **perpendicular** if $m_1 m_2 = -1$, and so $m_2 = \dfrac{-1}{m_1}$

We often say the slope of a perpendicular line has a slope that is the negative reciprocal

Example 9

What slope would be perpendicular to a line with:
A slope of 2?
A slope of -4?
A slope of $\dfrac{2}{3}$?

If the original line had slope 2, the perpendicular slope would be $m_2 = \dfrac{-1}{2}$

If the original line had slope -4, the perpendicular slope would be $m_2 = \dfrac{-1}{-4} = \dfrac{1}{4}$

If the original line had slope $\dfrac{2}{3}$, the perpendicular slope would be $m_2 = \dfrac{-1}{2/3} = \dfrac{-3}{2}$

Example 10

Find the equation of a line perpendicular to $f(x) = 6 + 3x$ and passing through the point (3, 0)

The original line has slope $m = 3$. The perpendicular line will have slope $m = \dfrac{-1}{3}$.

Using this and the given point, we can find the equation for the line.

$g(x) = b - \dfrac{1}{3}x$ then at (3, 0),

$0 = b - \dfrac{1}{3}(3)$

$b = 1$

The line we're looking for is $g(x) = 1 - \dfrac{1}{3}x$

Try it Now

3. Given the line $h(t) = -4 + 2t$ find a line that is a) Parallel and b) Perpendicular and both lines must pass through the point (0, 0)

Example 12

A line passes through the points (-2, 6) and (4, 5). Find the equation of a perpendicular line that passes through the point (4, 5).

From the two given points on the reference line, we can calculate the slope of that line:

$$m_1 = \frac{5-6}{4-(-2)} = \frac{-1}{6}$$

The perpendicular line will have slope

$$m_2 = \frac{-1}{-1/6} = 6$$

We can then solve for the vertical intercept to pass through the desired point:

$g(x) = b + 6x$ then at (4, 5),

$5 = b + 6(4)$

$b = -19$

Giving the line $g(x) = -19 + 6x$

Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Example 13

Find the intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$

Setting $h(t) = j(t)$,

$3t - 4 = 5 - t$

$4t = 9$

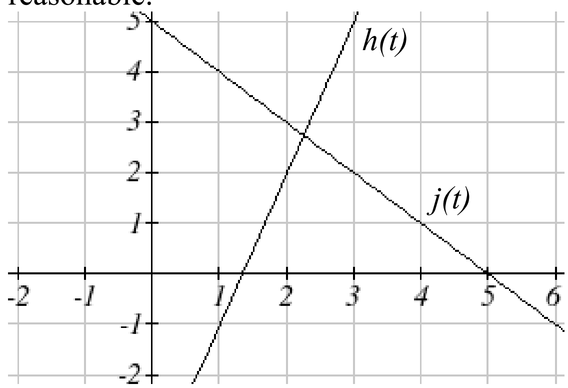
$t = \frac{9}{4}$

This tells us the lines intersect when the input is 9/4.

We can then find the output value of the intersection point by evaluating either function at this input

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} = \frac{11}{4}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$. Looking at the graph, this result seems reasonable.



Try it Now

4. Look at the graph in example 13 above and answer the following for the function *j(t)*:
a. Vertical intercept coordinates
b. Horizontal intercepts coordinates
c. Slope
d. Is *j(t)* parallel or perpendicular to *h(t)* (or neither)
e. Is *j(t)* an Increasing or Decreasing function (or neither)
f. Write a transformation description from the identity toolkit function *f(x) = x*

Finding the intersection allows us to answer other questions as well, such as discovering when one function is larger than another.

Example 14

Using the functions from the previous example, for what values of t is $h(t) > j(t)$

To answer this question, it is helpful first to know where the functions are equal, since that is the point where *h(t)* could switch from being greater to smaller than *j(t)* or vice-versa. From the previous example, we know the functions are equal at $t = \frac{9}{4}$. By examining the graph, we can see that *h(t)*, the function with positive slope, is going to be larger than the other function to the right of the intersection. So $h(t) > j(t)$ when

$$t > \frac{9}{4}$$

Important Topics of this Section

Methods for graphing linear functions
Another name for slope = rise/run
Horizontal intercepts (a,0)
Horizontal lines
Vertical lines
Parallel lines
Perpendicular lines
Intersecting lines

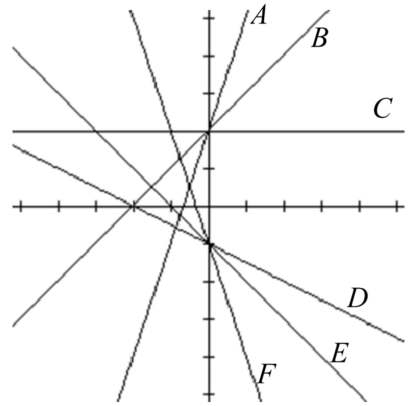
Try it Now Answers

1. (-3,7) found by starting at the vertical intercept, going up 2 units and 3 in the negative direction. You could have also answered, (-6, 9) or (-9, 11) etc…
2. Vertically stretched by a factor of 3, Vertically flipped (flipped over the x axis), Vertically shifted up by 6 units. $6-3x=0$ when $x=2$
3. Parallel $f(t) = 2t$; Perpendicular $g(t) = -1/2t$
4. Given $j(t) = 5-t$
   a. (0,5)
   b. (5,0)
   c. Slope -1
   d. Neither parallel nor perpendicular
   e. Decreasing function
   f. Given the identity function, perform a vertical flip (over the t axis) and shift up 5 units.

Section 2.2 Exercises

Match each linear equation with its graph

1. $f(x) = -x - 1$
2. $f(x) = -2x - 1$
3. $f(x) = -\frac{1}{2}x - 1$
4. $f(x) = 2$
5. $f(x) = 2 + x$
6. $f(x) = 3x + 2$



Sketch a line with the given features

7. An x-intercept of (-4, 0) and y-intercept of (0, -2)
8. An x-intercept of (-2, 0) and y-intercept of (0, 4)
9. A vertical intercept of (0, 7) and slope $-\frac{3}{2}$
10. A vertical intercept of (0, 3) and slope $\frac{2}{5}$
11. Passing through the points (-6,-2) and (6,-6)
12. Passing through the points (-3,-4) and (3,0)

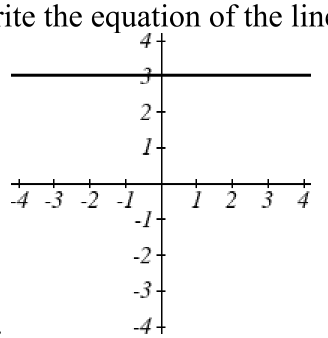
Sketch each equation

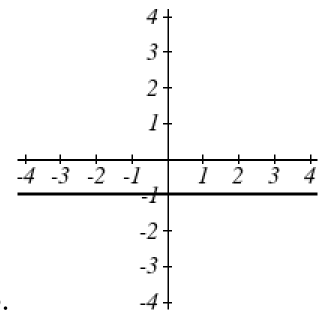
13. $f(x) = -2x - 1$
14. $g(x) = -3x + 2$
15. $h(x) = \frac{1}{3}x + 2$
16. $k(x) = \frac{2}{3}x - 3$
17. $k(t) = 3 + 2t$
18. $p(t) = -2 + 3t$
19. $x = 3$
20. $x = -2$
21. $r(x) = 4$
22. $q(x) = 3$

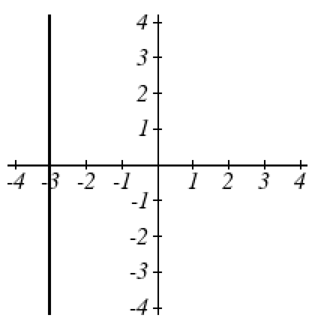
23. If $g(x)$ is the $f(x) = x$ after a vertical compression by $3/4$, a shift left by 2, and a shift down by 4
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

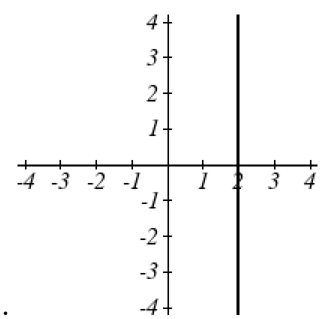
24. If $g(x)$ is the $f(x) = x$ after a vertical compression by $1/3$, a shift right by 1, and a shift up by 3
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

Write the equation of the line shown

25. 

26. 

27. 

28. 

Find the horizontal and vertical intercepts of each equation

29. $f(x) = -x + 2$
30. $g(x) = 2x + 4$
31. $h(x) = 3x - 5$
32. $k(x) = -5x + 1$
33. $-2x + 5y = 20$
34. $7x + 2y = 56$

Given below are descriptions of two lines. Find the slope of Line 1 and Line 2. Are each pair of lines parallel, perpendicular or neither?

35. Line 1: Passes through $(0,6)$ and $(3,-24)$
    Line 2: Passes through $(-1,19)$ and $(8,-71)$

36. Line 1: Passes through $(-8,-55)$ and $(10,89)$
    Line 2: Passes through $(9,-44)$ and $(4,-14)$

37. Line 1: Passes through $(2,3)$ and $(4,-1)$
    Line 2: Passes through $(6,3)$ and $(8,5)$

38. Line 1: Passes through $(1,7)$ and $(5,5)$
    Line 2: Passes through $(-1,-3)$ and $(1,1)$

39. Line 1: Passes through $(0,5)$ and $(3,3)$
    Line 2: Passes through $(1,-5)$ and $(3,-2)$

40. Line 1: Passes through $(2,5)$ and $(5,-1)$
    Line 2: Passes through $(-3,7)$ and $(3,-5)$

41. Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point (2,-12)

42. Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point (4,9)

43. Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point (-4,-1)

44. Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point (3,1)

45. Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$

46. Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$

47. Use algebra to find the point at which the line $f(x)=-\frac{4}{5}x+\frac{274}{25}$ intersects the line $h(x)=\frac{9}{4}x+\frac{73}{10}$

48. Use algebra to find the point at which the line $f(x)=\frac{7}{4}x+\frac{457}{60}$ intersects the line $g(x)=\frac{4}{3}x+\frac{31}{5}$

49. A car rental company offers two plans for renting a car.
Plan A: 30 dollars per day and 18 cents per mile
Plan B: 50 dollars per day with free unlimited mileage
For what range of miles will plan B save you money?

50. A cell phone company offers two data options for its prepaid phones
Pay per use: \$0.002 per Kilobyte (KB) used
Data Package: \$5 for 5 Megabytes (5120 Kilobytes) + \$0.002 per addition KB
Assuming you will use less than 5 Megabytes, for what range of use will the data package save you money?

51. Sketch an accurate picture of the line having equation $f(x)=2-\frac{1}{2}x$. Let c be an unknown constant. [UW]
   a. Find the point of intersection between the line you have graphed and the line $g(x)=1+cx$; your answer will be a point in the *xy* plane whose coordinates involve the unknown c.
   b. Find c so that the intersection point in (a) has x-coordinate 10.
   c. Find c so that the intersection point in (a) lies on the x-axis.

Section 2.3 Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem-solving strategies that we would use for any type of function:

Problem solving strategy

1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5) When needed, find a formula for the function.
6) Solve or evaluate using the formula you found for the desired quantities.
7) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Example 1

Emily saved up $3500 for her summer visit to Seattle. She anticipates spending $400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.
Output: M, money remaining, in dollars
Input: t, time, in weeks

Reading the problem, we identify two important values. The first, $3500, is the initial value for M. The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M(t) = 3500 - 400t$.

To find the horizontal intercept, we set the output to zero, and solve for the input:
$0 = 3500 - 400t$

$t = \frac{3500}{400} = 8.75$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be zero, interpreting this, we could say: Emily will have no money left after 8.75 weeks.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \le t \le 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \le t \le 8$, or $0 \le t \le 9$ if she went into debt to finish out the last week.

The range represents the set of output values and she starts with \$3500 and ends with \$0 after 8.75 weeks so the corresponding range is $0 \le M(t) \le 3500$.

If we limit the rental to whole weeks however, if she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500 - 400(8) =$ \$300 dollars left after 8 weeks, giving a range of $300 \le M(t) \le 3500$. If she wanted to stay the full 9 weeks she would be \$100 in debt giving a range of $-100 \le M(t) \le 3500$.

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable)

Example 2

Jamal is choosing between two moving companies. The first, U-haul, charges an up-front fee of \$20, then 59 cents a mile. The second, Budget, charges an up-front fee of \$16, then 63 cents a mile[3]. When will U-haul be the better choice for Jamal?

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

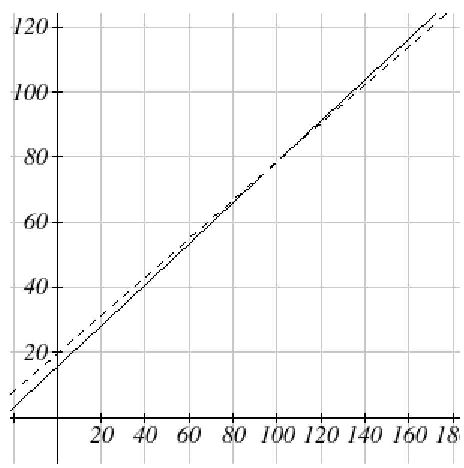
Input: *m*, miles driven
Outputs:
Y(m): cost, in dollars, for renting from U-haul
B(m): cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 a mile for U-haul, and \$0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the *Y(m)* function is smaller. Using the rates of change and initial charges, we can write the equations:

$Y(m) = 20 + 0.59m$

$B(m) = 16 + 0.63m$



These graphs are sketched to the right, with *Y(m)* drawn dashed.

To find the intersection, we set the equations equal and solve:

$Y(m) = B(m)$

$20 + 0.59m = 16 + 0.63m$

$4 = 0.04m$

$m = 100$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that *Y(m)* is growing at a slower rate, we can conclude that U-haul will be the cheaper price when more than 100 miles are driven.

[3] Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

Example 3

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,
a. Predict the population in 2013
b. When will the population reach 15000?

The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!
To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004:
Input: t, years since 2004
Output: $P(t)$, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point (0, 6200). Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving the point (5, 8100).

To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = 380 \text{ people per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t$$

To predict the population in 2013, we evaluate our function at $t = 9$

$$P(9) = 6200 + 380(9) = 9620$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$15000 = 6200 + 380t$$

$$8800 = 380t$$

$$t \approx 23.158$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

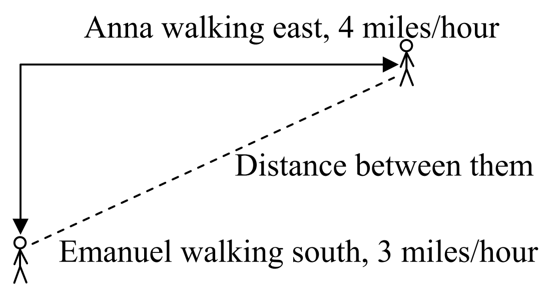
Example 4

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio with a range of 2 miles. How long after they start walking will they fall out of radio contact?

In essence, we can partially answer this question by saying; they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question: how long will it take them to be 2 miles apart?

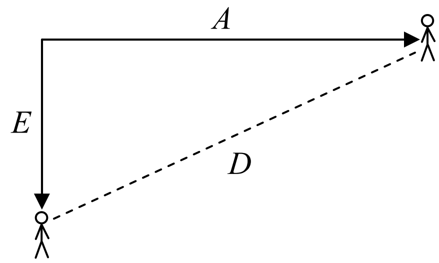
In this problem, our changing quantities are time and the two peoples' positions, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define
Input: t, time in hours.

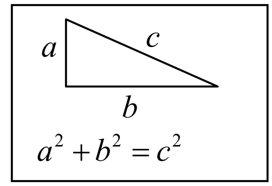
Since it is not obvious how to define our output variables, we'll start by drawing a picture.



Because of the complexity of this question, it may be helpful to introduce some intermediary variables. These are quantities that we aren't directly interested in, but seem important to the problem. For this problem, Anna's and Emanuel's distances from the starting point seem important. To notate these, we are going to define a coordinate system, putting the "starting point" at the intersection where they both started, then we're going to introduce a variable, A, to represent Anna's position, and define it to be a measurement from the starting point, in the eastward direction. Likewise, we'll introduce a variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system we specified both the origin, or starting point, of the measurement, as well as the direction of measure.

While we're at it, we'll define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the picture is often helpful: Looking at the variables on the picture, we remember we need to know how long it takes for D, the distance between them to equal 2 miles.





Seeing this picture we remember that in order to find the distance between the two, we can use the Pythagorean theorem, a property of right triangles.

From here, we can now look back at the problem for relevant information. Anna is walking 4 miles per hour, and Emanuel is walking 3 miles per hour, which are rates of change. Using those, we can write formulas for the distance each has walked.

They both start at the same intersection and so when t = 0, the distance travelled by each person should also be 0, so given the rate for each, and the initial value for each we get:

$$A(t) = 4t$$
$$E(t) = 3t$$

Using the Pythagorean theorem we get:

$$D(t)^2 = A(t)^2 + E(t)^2$$
$$D(t)^2 = (4t)^2 + (3t)^2 = 16t^2 + 9t^2 = 25t^2$$
$$D(t) = \sqrt{25t^2} = 5t$$

Interestingly, the distance between them is also a linear function. Using it, we can now answer the question of when the distance between them will reach 2 miles:

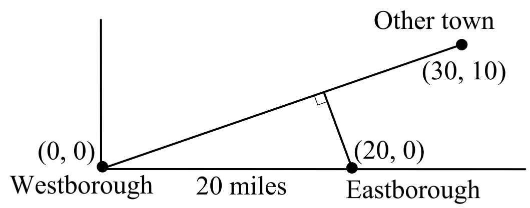
$$D(t) = 2$$
$$5t = 2$$
$$t = \frac{2}{5} = 0.4$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Example 5

There is currently a straight road leading from the town of Westborough to a town 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

It might help here to draw a picture of the situation. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts the other town at coordinates (30, 10), and Eastborough at (20, 0)



Using this point along with the origin, we can find the slope of the line from Westborough to the other town: $m = \frac{10-0}{30-0} = \frac{1}{3}$. This gives the equation of the road from Westborough to the other town to be $W(x) = \frac{1}{3}x$.

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Since the town of Eastborough is at the point (20, 0), we can find the equation:

$E(x) = -3x + b$ plug in the point (20, 0)

$0 = -3(20) + b$

$b = 60$

$E(x) = -3x + 60$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$\frac{1}{3}x = -3x + 60$

$\frac{10}{3}x = 60$

$10x = 180$

$x = 18$ Substituting this back into *W(x)*

$y = W(18) = \frac{1}{3}(18) = 6$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction:

$dist = \sqrt{(18-0)^2 + (6-0)^2} \approx 18.934$ miles

Important Topics of this Section

The problem solving process

1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5) When needed, find a formula for the function.
6) Solve or evaluate using the formula you found for the desired quantities.
7) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Section 2.3 Exercises

1. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.
    a. How much did the population grow between the year 2004 and 2008?
    b. How long did it take the population to grow from 1001 students to 1697 students?
    c. What is the average population growth per year?
    d. What was the population in the year 2000?
    e. Find an equation for the population, P, of the school t years after 2000.
    f. Using your equation, predict the population of the school in 2011.

2. In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.
    a. How much did the population grow between the year 2003 and 2007?
    b. How long did it take the population to grow from 1431 people to 2134?
    c. What is the average population growth per year?
    d. What was the population in the year 2000?
    e. Find an equation for the population, P, of the town t years after 2000.
    f. Using your equation, predict the population of the town in 2014.

3. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be \$71.50. If the customer uses 720 minutes, the monthly cost will be \$118.
    a. Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
    b. Interpret the slope and vertical intercept of the equation.
    c. Use your equation to find the total monthly cost if 687 minutes are used.

4. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be \$11.20. If the customer uses 130 MB, the monthly cost will be \$17.80.
    a. Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
    b. Interpret the slope and vertical intercept of the equation.
    c. Use your equation to find the total monthly cost if 250 MB are used.

5. In 1991, the moose population in a park was measured to be 4360. By 1999, the population was measured again to be 5880. If the population continues to change linearly,
    a. Find a formula for the moose population, P.
    b. What does your model predict the moose population to be in 2003?

6. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. If the population continues to change linearly,
    a. Find a formula for the owl population, P.
    b. What does your model predict the owl population to be in 2012?

7. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010, and is being depleted by about 2.1 billion cubic feet each year.
    a. Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
    b. In 2015, what will the helium reserves be?
    c. If the rate of depletion isn't change, when will the Federal Helium Reserve be depleted?

8. Suppose the world's current oil reserves are 1820 billion barrels. If, on average, the total reserves is decreasing by 25 billion barrels of oil each year:
    a. Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
    b. Seven years from now, what will the oil reserves be?
    c. If the rate of depletion isn't change, when will the world's oil reserves be depleted?

9. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of \$19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

10. You are choosing between two different window washing companies. The first charges \$5 per window. The second charges a base fee of \$40 plus \$3 per window. How many windows would you need to have for the second company to be preferable?

11. When hired at a new job selling jewelry, you are given two pay options:
    Option A: Base salary of \$17,000 a year, with a commission of 12% of your sales
    Option B: Base salary of \$20,000 a year, with a commission of 5% of your sales
    How much jewelry would you need to sell for option A to produce a larger income?

12. When hired at a new job selling electronics, you are given two pay options:
Option A: Base salary of $14,000 a year, with a commission of 10% of your sales
Option B: Base salary of $19,000 a year, with a commission of 4% of your sales
How much electronics would you need to sell for option A to produce a larger income?

13. Find the area of a triangle bounded by the *y* axis, the line $f(x)=9-\frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

14. Find the area of a triangle bounded by the *x* axis, the line $f(x)=12-\frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

15. Find the area of a parallelogram bounded by the *y* axis, the line $x=3$, the line $f(x)=1+2x$, and the line parallel to $f(x)$ passing through (2, 7)

16. Find the area of a parallelogram bounded by the *x* axis, the line $g(x)=2$, the line $f(x)=3x$, and the line parallel to $f(x)$ passing through (6, 1)

17. If $b>0$ and $m<0$, then the line $f(x)=b+mx$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of *m* and *b*. [UW]

18. Find the value of *m* so the lines $f(x)=mx+5$ and $g(x)=x$ and the *y*-axis form a triangle with an area of 10. [UW]

19. The median home value in Mississippi and Hawaii (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Mississippi	Hawaii
1950	25200	74400
2000	71400	272700

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Mississippi in 2010?
c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

20. The median home value in Indiana and Alabama (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Indiana	Alabama
1950	37700	27100
2000	94300	85100

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Indiana in 2010?
c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

21. Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome. On her trip, how close does Pam get to Paris? [UW]

22. You're flying from Joint Base Lewis-McChord (JBLM) to an undisclosed location 226 km south and 230 km east. Mt. Rainier is located approximately 56 km east and 40 km south of JBLM. If you are flying at a constant speed of 800 km/hr, how long after you depart JBLM will you be the closest to Mt. Rainier?

Section 2.4 Fitting Linear Models to Data

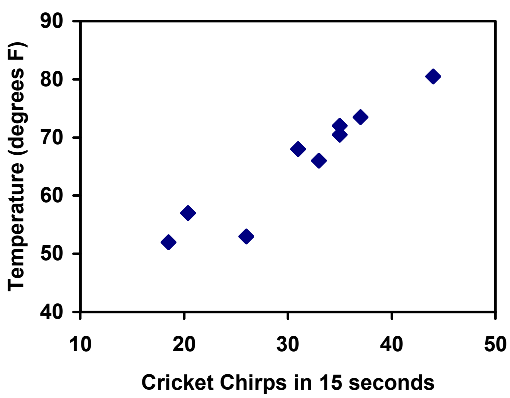
In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

Example 1

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit[4]. Plot this data, and determine whether the data appears to be linearly related.

chirps	44	35	20.4	33	31	35	18.5	37	26
Temp	80.5	70.5	57	66	68	72	52	73.5	53

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.



The simplest way to find an equation to approximate this data is to try to "eyeball" a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

[4] Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

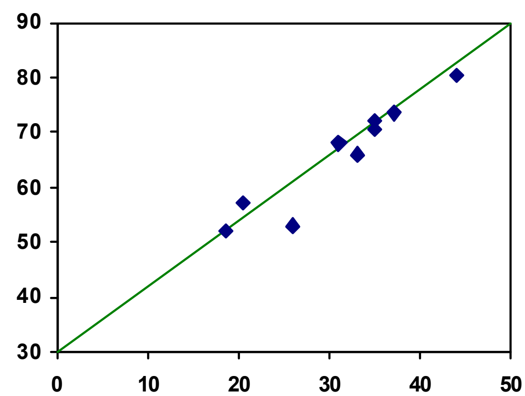
Flashback

1. a. What descriptive variables would you choose to represent Temperature & Chirps?
   b. Which variable is the independent variable and which is the dependent variable?
   c. Based on this data and the graph, what is a reasonable domain & range?
   d. Based on the data alone, is this function one-to-one, explain?

Example 2

Using the table of values from the previous example, find a linear function that fits the data by "eyeballing" a line that seems to fit.

On a graph, we could try sketching in a line. The scale on the axes has been adjusted to including the vertical axis in the graph.



Using the starting and ending points of our "hand drawn" line, points (0, 30) and (50, 90), this graph has a slope of $m = \frac{60}{50} = 1.2$ and a vertical intercept at 30, giving an equation of

$T(c) = 30 + 1.2c$
where *c* is the number of chirps in 15 seconds, and *T(c)* is the temperature in degrees Fahrenheit.

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.

Interpolation and Extrapolation

Interpolation: When we predict a value inside the domain and range of the data
Extrapolation: When we predict a value outside the domain and range of the data

For the Temperature as a function of chirps in our hand drawn model above:

Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.

Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

Example 3

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:
$T(30) = 30 + 1.2(30) = 66$ degrees.
Based on the data we have, this value seems reasonable.

The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c$$

$$10 = 1.2c$$

$$c \approx 8.33$$

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**.

Try it Now

What temperature would you predict if you counted 20 chirps in 15 seconds?

Fitting Lines with Technology

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[5]. This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators[6].

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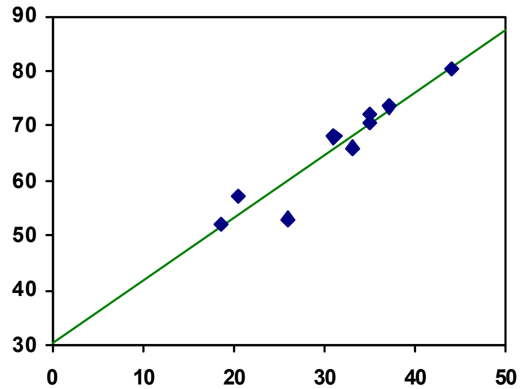
[5] Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.
[6] For example, http://www.shodor.org/unchem/math/lls/leastsq.html

Example 4

Find the least-squares regression line using the cricket chirp data from above.



Using the cricket chirp data from earlier, with technology we obtain the equation:
$T(c) = 30.281 + 1.143c$

Notice that this line is quite similar to the equation we "eyeballed", but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:
$T(30) = 30.281 + 1.143(30) = 64.571 \approx 64.6$ degrees.

Most calculators and computer software will also provide you with the **correlation coefficient**, a measure of how closely the line fits the data.

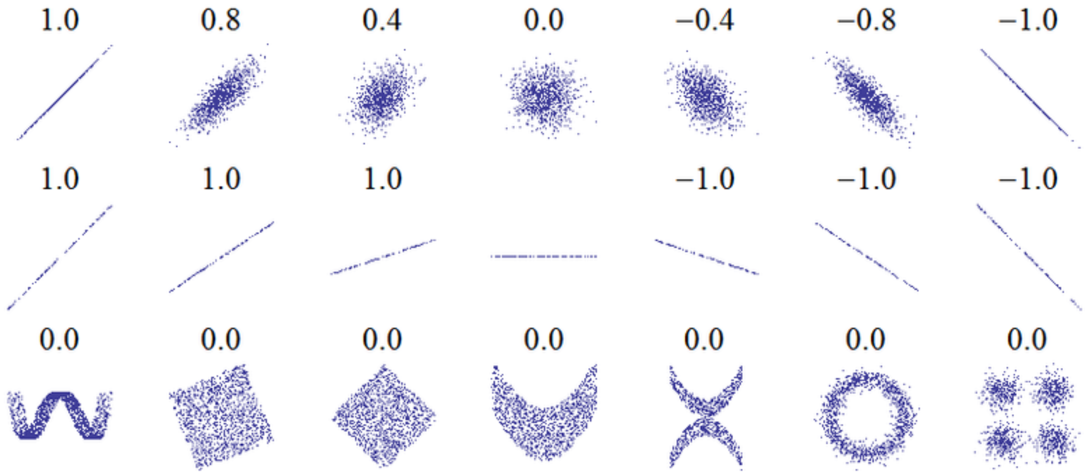
Correlation Coefficient

The **correlation coefficient** is a value, r, between -1 and 1.
$r > 0$ suggests a positive (increasing) relationship
$r < 0$ suggests a negative (decreasing) relationship
The closer the value is to 0, the more scattered the data
The closer the value is to 1 or -1, the less scattered the data is

The correlation coefficient provides an easy way to get some idea of how close to a line the data falls.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of r and the graph of the data, here are some large data sets with their correlation coefficients:

Examples of Correlation Coefficient Values



[7]

Example 5

Calculate the correlation coefficient for our cricket data.

Because the data appears to follow a linear pattern, we can use technology to calculate $r = 0.9509$. Since this value is very close to 1, it suggests a strong increasing linear relationship.

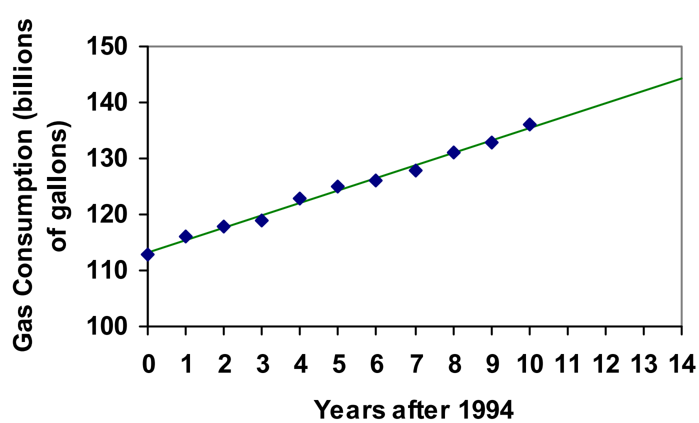
Example 6

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.[8] Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billion of gallons)	113	116	118	119	123	125	126	128	131	133	136

To make things simpler, a new input variable is introduced, *t*, representing years since 1994.

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.



[7] http://en.wikipedia.org/wiki/File:Correlation_examples.png
[8] http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

The least-squares regression equation is:
$C(t) = 113.318 + 2.209t$.

Using this to predict consumption in 2008 (t = 14),
$C(14) = 113.318 + 2.209(14) = 144.244$ billions of gallons

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

Try it Now

2. Use the model created by technology in example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Important Topics of this Section

Fitting linear models to data by hand
Fitting linear models to data using technology
Interpolation
Extrapolation
Correlation coefficient

Flashback Answers

1. a. T = Temperature, C = Chirps (answers may vary)
   b. Independent (Chirps) , Dependent (Temperature)
   c. Reasonable Domain (18.5, 44) , Reasonable Range (52, 80.5) (answers may vary)
   d. NO, it is not one-to-one, there are two different output values for 35 chirps.

Try it Now Answers

1. 54 degrees Fahrenheit
2. 150.871 billions of gallons, extrapolation

Section 2.4 Exercises

1. The following is data for the first and second Quiz scores for 8 students in a class. Plot the points, then sketch a line that best fits the data.

First Quiz	11	20	24	25	33	42	46	49
Second Quiz	10	16	23	28	30	39	40	49

2. Eight students were asked to estimate their score on a 10 point quiz. Their estimated and actual scores are given. Plot the points, then sketch a line that best fits the data.

Predicted	5	7	6	8	10	9	10	7
Actual	6	6	7	8	9	9	10	6

Based on each set of data given, calculate the regression line using your calculator or other technology tool, and determine the correlation coefficient.

3.

x	*y*
5	4
7	12
10	17
12	22
15	24

4.

x	*y*
8	23
15	41
26	53
31	72
56	103

5.

x	*y*
3	21.9
4	22.22
5	22.74
6	22.26
7	20.78
8	17.6
9	16.52
10	18.54
11	15.76
12	13.68
13	14.1
14	14.02
15	11.94
16	12.76
17	11.28
18	9.1

6.

x	*y*
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8
14	24.7
15	22
16	20.1
17	19.8
18	16.8

7. A regression was run to determine if there is a relationship between hours of TV watched per day (x) and number of situps a person can do (y). The results of the regression are given below. Use this to predict the number of situps a person who watches 11 hours of TV can do.

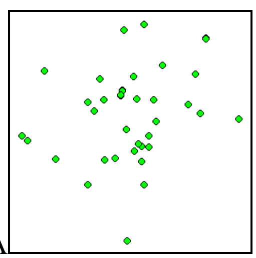
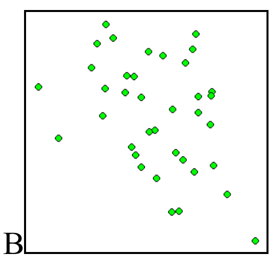
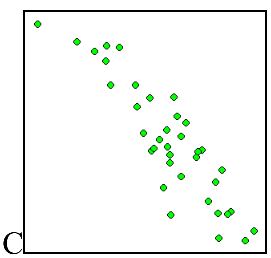
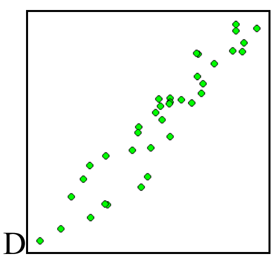
```
y=ax+b
a=-1.341
b=32.234
r²=0.803
r=-0.896
```

8. A regression was run to determine if there is a relationship between the diameter of a tree (x, in inches) and the tree's age (y, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

```
y=ax+b
a=6.301
b=-1.044
r²=0.940
r=-0.970
```

Match each scatterplot shown below with one of the four specified correlations.

9. $r = 0.95$ 10. $r = -0.89$ 11. $r = 0.26$ 12. $r = -0.39$

A 
B 
C 
D 

13. The US census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will the percentage exceed 35%?

Year	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Percent Graduates	21.3	21.4	22.2	23.6	24.4	25.6	26.7	27.7	28	29.4

14. The US import of wine (in hectoliters) for several years if given below. Determine if the trend appears linear. If so and the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	1992	1994	1996	1998	2000	2002	2004	2006	2008	2009
Imports	2665	2688	3565	4129	4584	5655	6549	7950	8487	9462

Section 2.5 Absolute Value Functions

So far in this chapter we have been studying the behavior of linear functions. The Absolute Value Functions is a piecewise defined function made up of two linear functions. The name, Absolute Value Function, should be familiar to you from Section 1.2. In its basic form $f(x) = |x|$ it is one of our toolkit functions.

Absolute Value Function

The absolute value function can be defined as

$$f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

The absolute value function is commonly used to determine the distance between two numbers on the number line. Given two values *a* and *b*, then $|a - b|$ will give the distance, a positive quantity, between these values, regardless of which value is larger.

Example 1

Describe all values, *x*, within a distance of 4 from the number 5.

We want the distance between *x* and 5 to be less than or equal to 4. The distance can be represented using the absolute value, giving the expression

$|x - 5| \leq 4$

Example 2

A survey poll reports in 2010 reported 78% of Americans believe that people who are gay should be able to serve in the US military, with a reported margin of error of 3%[9]. The margin of error tells us how far off the actual value could be from the survey value[10]. Express the set of possible values using absolute values.

Since we want the size of the difference between the actual percentage, *p*, and the reported percentage to be less than 3%,

$|p - 78| \leq 3$

---

[9] http://www.pollingreport.com/civil.htm, retrieved August 4, 2010

[10] Technically, margin of error usually means that the surveyors are 95% confident that actual value falls within this range.

Try it Now

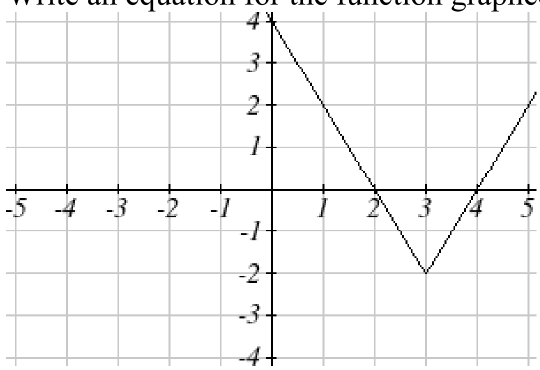
1. Students who score within 20 points of 80 will pass the test. Write this as a distance from 80 using the absolute value notation.

Important Features

The most significant feature of the absolute value graph is the corner point where the graph changes direction. When finding the equation for a transformed absolute value function, this point is very helpful for determining the horizontal and vertical shifts.

Example 3

Write an equation for the function graphed below.



The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 and down 2 from the basic toolkit function. We might also notice that the graph appears stretched, since the linear portions have slopes of 2 and -2. From this information we can write the equation:

$f(x) = 2|x-3|-2$, treating the stretch as a vertical stretch

$f(x) = |2(x-3)|-2$, treating the stretch as a horizontal compression

Note that these equations are algebraically equivalent – the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch/compression.

If you had not been able to determine the stretch based on the slopes of the lines, you can solve for the stretch factor by putting in a known pair of values for x and *f(x)*

$f(x) = a|x-3|-2$ Now substituting in the point (1, 2)

$2 = a|1-3|-2$

$4 = 2a$

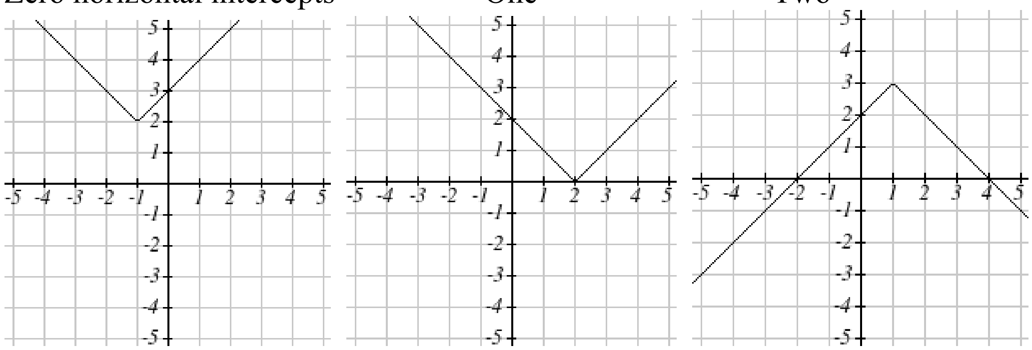
$a = 2$

Try it Now

2. Given the description of the transformed absolute value function write the equation. The absolute value function is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units,

The graph of an absolute value function will have a vertical intercept, when the input is zero. The graph may or may not have horizontal intercepts, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to have zero, one, or two horizontal intercepts.

Zero horizontal intercepts One Two



To find the horizontal intercepts, we will need to solve an equation involving an absolute value.

Notice that the absolute value function is not one-to-one, so typically inverses of absolute value functions are not discussed.

Solving Absolute Value Equations

To solve an equation like $8 = |2x - 6|$, we can notice that the absolute value will be equal to eight if the quantity *inside* the absolute value were 8 or -8. This leads to two different equations we can solve independently:

$2x - 6 = 8$ or	$2x - 6 = -8$
$2x = 14$	$2x = -2$
$x = 7$	$x = -1$

Solutions to Absolute Value Equations

An equation of the form $|A| = B$, with $B \ge 0$, will have solutions when

$A = B$ or $A = -B$

Example 4

Find the horizontal intercepts of the graph of $f(x) = |4x+1| - 7$

The horizontal intercepts will occur when $f(x) = 0$. Solving,

$0 = \|4x+1\| - 7$		Isolate the absolute value on one side of the equation
$7 = \|4x+1\|$		Now we can break this into two separate equations:
$7 = 4x+1$		$-7 = 4x+1$
$6 = 4x$	or	$-8 = 4x$
$x = \frac{6}{4} = \frac{3}{2}$		$x = \frac{-8}{4} = -2$

The graph has two horizontal intercepts, at $x = \frac{3}{2}$ and x = -2

Example 5

Solve $1 = 4|x-2| + 2$

Isolating the absolute value on one side the equation,

$1 = 4|x-2| + 2$

$-1 = 4|x-2|$

$-\frac{1}{4} = |x-2|$

At this point, we notice that this equation has no solutions – the absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value.

Try it Now

3. Find the horizontal & vertical intercepts for the function $f(x) = -|x+2| + 3$

Solving Absolute Value Inequalities

When absolute value inequalities are written to describe a set of values, like the inequality $|x-5| \le 4$ we wrote earlier, it is sometimes desirable to express this set of values without the absolute value, either using inequalities, or using interval notation.

We will explore two approaches to solving absolute value inequalities:

1) Using the graph
2) Using test values

Example 6

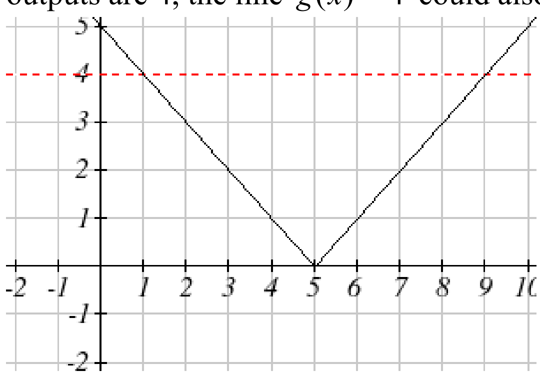
Solve $|x-5| \le 4$

With both approaches, we will need to know first where the corresponding *equality* is true. In this case we first will find where $|x-5| = 4$. We do this because the absolute value is a nice friendly function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5| = 4$,

$x - 5 = 4$ or $x - 5 = -4$

$x = 9$ $\qquad$ $x = 1$

To use a graph, we can sketch the function $f(x) = |x-5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched.



On the graph, we can see that indeed the output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$. Based on the shape of the graph, we can determine the absolute value is less than or equal to 4 between these two points, when $1 \le x \le 9$. In interval notation, this would be the interval [1,9].

As an alternative to graphing, after determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can only change from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals: $x<1$, $1<x<9$, and $x>9$. To determine when the function is less than 4, we could pick a value in each interval and see if the output is less than or greater than 4.

Interval	Test x	$f(x)$	<4 or >4?
$x<1$	0	$\|0-5\|=5$	greater
$1<x<9$	6	$\|6-5\|=1$	less
$x>9$	11	$\|11-5\|=6$	greater

Since the only interval in which the output at the test value is less than 4, we can conclude the solution to $|x-5|\le 4$ is $1\le x\le 9$.

Example 7

Given the function $f(x)=-\frac{1}{2}|4x-5|+3$, determine for what x values the function values are negative.

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3<0$. We begin by isolating the absolute value:

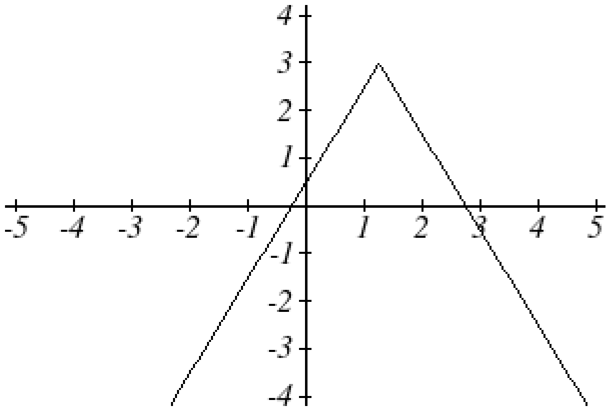
$-\frac{1}{2}|4x-5|<-3$ when we multiply both sides by -2, it reverses the inequality

$|4x-5|>6$

Next we solve for the equality $|4x-5|=6$

$$
\begin{array}{lcl}
4x-5=6 & & 4x-5=-6 \\
4x=11 & \text{or} & 4x=-1 \\
x=\frac{11}{4} & & x=\frac{-1}{4}
\end{array}
$$

We can now either pick test values or sketch a graph of the function to determine on which intervals the original function value are negative. Notice that it is not even really important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=\frac{-1}{4}$ and $x=\frac{11}{4}$, and that the graph has been reflected vertically.



From the graph of the function, we can see the function values are negative to the left of the first horizontal intercept at $x = \frac{-1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality:

$$x < \frac{-1}{4} \quad or \quad x > \frac{11}{4}$$

In interval notation, this would be $\left(-\infty, \frac{-1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$

Try it Now

4. Solve $-2|k-4| \le -6$

Important Topics of this Section

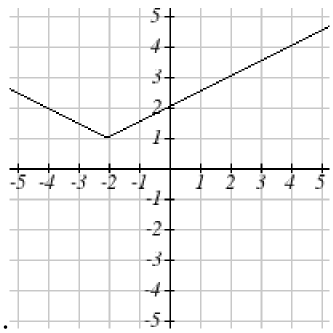
The properties of the absolute value function
Solving absolute value equations
Finding intercepts
Solving absolute value inequalities

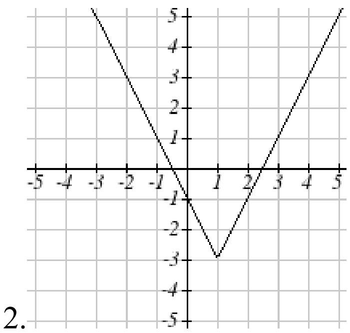
Try it Now Answers

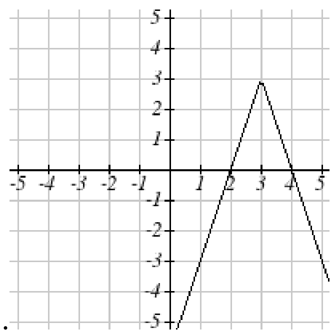
1. Using the variable p, for passing, $|p-80| \le 20$
2. $f(x) = -|x+2| + 3$
3. $f(0) = 1$, so the vertical intercept is at (0,1). $f(x) = 0$ when $x = -5$ and $x = 1$ so the horizontal intercepts are at (-5,0) & (1,0)
4. $k < 1$ or $k > 7$; in interval notation this would be $(-\infty, 1) \cup (7, \infty)$

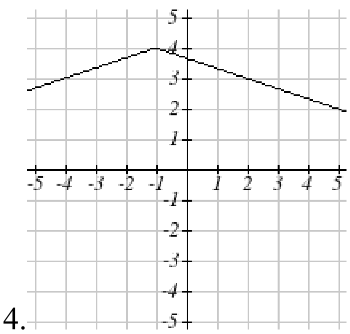
Section 2.5 Exercises

Write an equation for each transformation of $f(x) = |x|$

1. 

2. 

3. 

4. 

Sketch a graph of each function

5. $f(x) = -|x-1|-1$

6. $f\left(x\right) = -\left|x+3\right|+4$

7. $f\left(x\right) = 2\left|x+3\right|+1$

8. $f\left(x\right) = 3\left|x-2\right|-3$

9. $f\left(x\right) = \left|2x-4\right|-3$

10. $f\left(x\right) = \left|3x+9\right|+2$

Solve each the equation

11. $|5x-2| = 11$

12. $|4x+2| = 15$

13. $2|4-x| = 7$

14. $3|5-x| = 5$

15. $3\left|x+1\right|-4 = -2$

16. $5\left|x-4\right|-7 = 2$

Find the horizontal and vertical intercepts of each function

17. $f(x) = 2|x+1| - 10$

18. $f(x) = 4|x-3| + 4$

19. $f(x) = -3|x-2| - 1$

20. $f(x) = -2|x+1| + 6$

Solve each inequality

21. $|x+5| < 6$

22. $|x-3| < 7$

23. $|x-2| \geq 3$

24. $|x+4| \geq 2$

25. $|3x+9| < 4$

26. $|2x-9| \leq 8$

Chapter 3: Polynomial and Rational Functions

Section 3.1 Power Functions & Polynomial Functions

A square is cut out of cardboard, with each side having some length *L*. If we wanted to write a function for the area of the square, with *L* as the input, and the area as output, you may recall that area can be found by multiplying the length times the width. Since our shape is a square, the length & the width are the same, giving the formula:
$A(L) = L \cdot L = L^2$

Likewise, if we wanted a function for the volume of a cube with each side having some length *L*, you may recall that volume can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:
$V(L) = L \cdot L \cdot L = L^3$

These two functions are examples of **power functions**; functions that are some power of the variable.

Power Function

A **power function** is a function that can be represented in the form
$f(x) = x^p$
Where the base is the variable and the exponent, *p*, is a number.

Example 1

Which of our toolkit functions are power functions?

The constant and identity functions are power functions, since they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.
The quadratic and cubic functions are both power functions with whole number powers: $f(x) = x^2$ and $f(x) = x^3$.

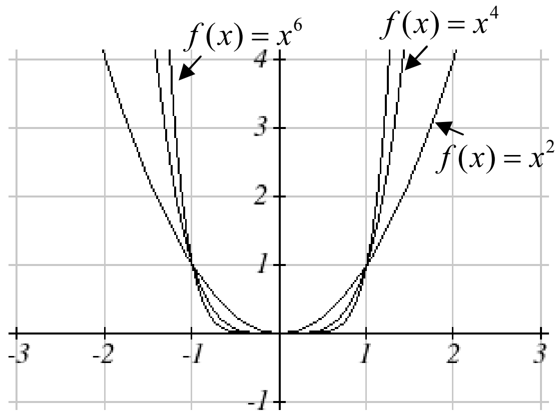
The rational functions are both power functions with negative whole number powers since they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are both power functions with fractional powers since they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Try it Now

1. What point(s) do the toolkit power functions have in common?

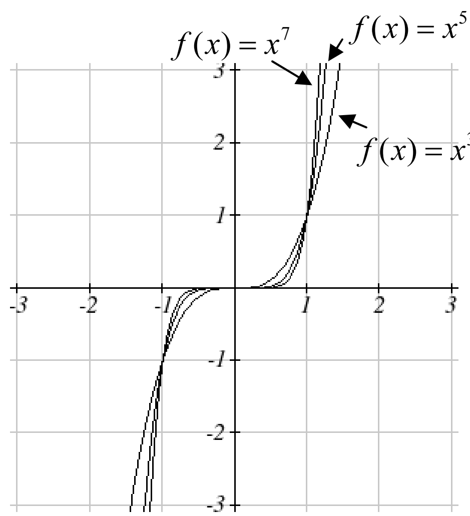
Characteristics of Power Functions



Shown to the right are the graphs of $f(x)=x^2$, $f(x)=x^4$, and $f(x)=x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to the quadratic toolkit, but as the power increases the graphs flatten somewhat near the origin, and grow faster as the input increases.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is ∞, and $-\infty$ for negative infinity. When we say that "*x* approaches infinity", which can be symbolically written as $x \to \infty$, we are describing a behavior – we are saying that *x* is getting large in the positive direction.

With the even power function, as the input becomes large in either the positive or negative directions, the output values become very large positive numbers. Equivalently, we could describe this by saying that as *x* approaches positive or negative infinity, the *f(x)* values approach positive infinity. In symbolic form, we could write: as $x \to \pm\infty$, $f(x) \to \infty$.



Shown here are the graphs of $f(x)=x^3$, $f(x)=x^5$, and $f(x)=x^7$, all odd whole number powers. Notice all these graphs look similar to the cubic toolkit, but again as the power increases the graphs flatten near the origin and grow faster as the input increases.

For these odd power functions, as *x* approaches negative infinity, *f(x)* approaches negative infinity. As *x* approaches positive infinity, *f(x)* approaches positive infinity. In symbolic form we write: as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Ling Run Behavior

The behavior of the graph of a function as the input takes on large negative values ($x \to -\infty$) and large positive values ($x \to \infty$) as is referred to as the **long run behavior** of the function.

Example 2

Describe the long run behavior of the graph of $f(x) = x^8$.

Since $f(x) = x^8$ has a whole, even power, we would expect this function to behave somewhat like the quadratic function. As the input gets large positive or negative, we would expect the output to grow in the positive direction. In symbolic form, as $x \to \pm\infty$, $f(x) \to \infty$.

Example 3

Describe the long run behavior of the graph of $f(x) = -x^9$

Since this function has a whole odd power, we would expect it to behave somewhat like the cubic function. The negative in front of the function will cause a vertical reflection, so as the inputs grow large positive, the outputs will grow large in the negative direction, and as the inputs grow large negative, the outputs will grow large in the positive direction. In symbolic form, for the long run behavior we would write: as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$.

You may use words or symbols to describe the long run behavior of these functions.

Try it Now

2. Describe in words and symbols the long run behavior of $f(x) = -x^4$

Treatment of the rational and radical forms of power functions will be saved for later.

Polynomials

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick roughly in a circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. If we wanted to write a formula for the area covered by the oil slick, we could do so by composing two functions together. The first is a formula for the radius, *r*, of the spill, which depends on the number of weeks, *w*, that have passed. Hopefully you recognized that this relationship is linear:
$r(w) = 24 + 8w$

We can combine this with the formula for the area, *A*, of a circle:
$A(r) = \pi r^2$

Composing these functions gives a formula for the area in terms of weeks:
$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2$

Multiplying this out gives the formula
$A(w) = 576\pi + 384\pi w + 64\pi w^2$

This formula is an example of a **polynomial**. A polynomial is simply the sum of terms consisting of transformed power functions with positive whole number powers.

Terminology of Polynomial Functions

A **polynomial** is function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Each of the a_i constants are called **coefficients** and can be positive, negative, whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, any $a_i x^i$. Each individual term is a transformed power function

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable; the term with the highest degree.

The **leading coefficient** is the coefficient on the leading term.

Because of the definition of the leading term we often rearrange polynomials so that the powers are descending and the parts are easier to determine.
$f(x) = a_n x^n + \ldots.. + a_2 x^2 + a_1 x + a_0$

Example 4

Identify the degree, leading term, and leading coefficient of these polynomials:
$f(x) = 3 + 2x^2 - 4x^3$
$g(t) = 5t^5 - 2t^3 + 7t$
$h(p) = 6p - p^3 - 2$

For the function *f(x)*, the degree is 3, the highest power on *x*. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For *g(t)*, the degree is 5, the leading term is $5t^5$, and the leading coefficient is 5.

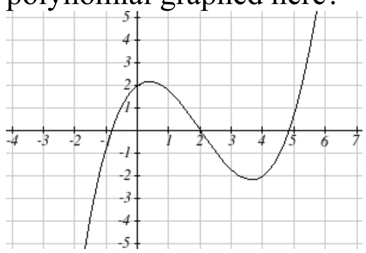
For *h(p)*, the degree is 3, the leading term is $-p^3$, so the leading coefficient is -1.

Long Run Behavior of Polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

Example 5

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Since the graph grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to -\infty$.

In words we could say that as *x* values approach infinity, the function values approach infinity, and as *x* values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd.

Try it Now

3. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ use your algebra skills write the function in polynomial form and determine the leading term, degree, and long run behavior of the function.

Short Run Behavior

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at a_0, or the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Example 6

Given the polynomial function $f(x)=(x-2)(x+1)(x-4)$, given in factored form for your convenience, determine the vertical and horizontal intercepts.

The vertical intercept occurs when the input is zero.
$f(0)=(0-2)(0+1)(0-4)=8$.

The graph crosses the vertical axis at the point (0, 8)

The horizontal intercepts occur when the output is zero.
$0=(x-2)(x+1)(x-4)$ when x = 2, -1, or 4

The graph crosses the horizontal axis at the points (2, 0), (-1, 0), and (4, 0)

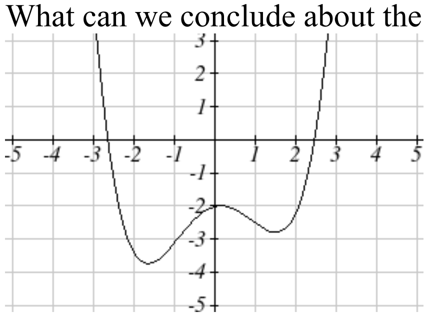
Notice that the polynomial in the previous example, which would be degree three if multiplied out, had three horizontal intercepts and two turning points - places where the graph changes direction. We will make a general statement here without justification at this time – the reasons will become clear later in this chapter.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have:
At most n horizontal intercepts. An odd degree polynomial will always have at least one.
At most n-1 turning points

Example 7

What can we conclude about the graph of the polynomial shown here?



Based on the long run behavior, with the graph becoming large positive on both ends of the graph, we can determine that this is the graph of an even degree polynomial. The graph has 2 horizontal intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4, so it is probably a fourth degree polynomial.

Try it Now

4. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ determine the short run behavior.

Important Topics of this Section

Power Functions
Polynomials
Coefficients
Leading coefficient
Term
Leading Term
Degree of a polynomial
Long run behavior
Short run behavior

Try it Now Answers

1. (0, 0) and (1, 1) are common to all power functions
2. As *x* approaches positive and negative infinity, *f(x)* approaches negative infinity: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the vertical flip.
3. The leading term is $0.2x^3$, so it is a degree 3 polynomial, as *x* approaches infinity (or gets very large in the positive direction) *f(x)* approaches infinity, and as *x* approaches negative infinity (or gets very large in the negative direction) *f(x)* approaches negative infinity. (Basically the long run behavior is the same as the cubic function)
4. Horizontal intercepts are (2, 0) (-1, 0) and (5, 0), the vertical intercept is (0, 2) and there are 2 turns in the graph.

Section 3.1 Exercises

Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

1. $f(x) = x^4$ 2. $f(x) = x^6$ 3. $f(x) = x^3$ 4. $f(x) = x^5$
5. $f(x) = -x^2$ 6. $f(x) = -x^4$ 7. $f(x) = -x^7$ 8. $f(x) = -x^9$

Find the degree and leading coefficient of each polynomial

9. $4x^7$
10. $5x^6$
11. $5 - x^2$
12. $6 + 3x - 4x^3$
13. $-2x^4 - 3x^2 + x - 1$
14. $6x^5 - 2x^4 + x^2 + 3$
15. $(2x+3)(x-4)(3x+1)$
16. $(3x+1)(x+1)(4x+3)$

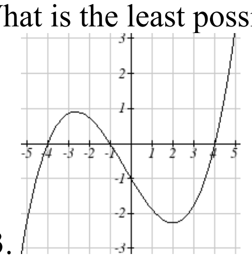
Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

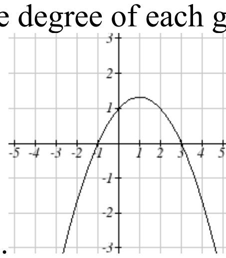
17. $-2x^4 - 3x^2 + x - 1$
18. $6x^5 - 2x^4 + x^2 + 3$
19. $3x^2 + x - 2$
20. $-2x^3 + x^2 - x + 3$

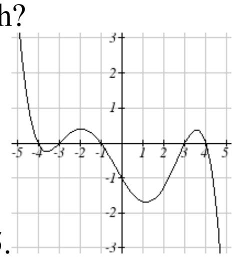
21. What is the maximum number of *x*-intercepts and turning points for a polynomial of degree 5?

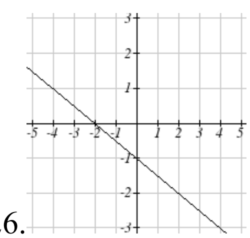
22. What is the maximum number of *x*-intercepts and turning points for a polynomial of degree 8?

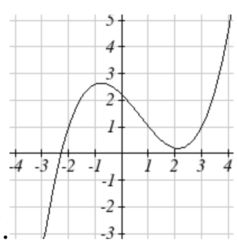
What is the least possible degree of each graph?

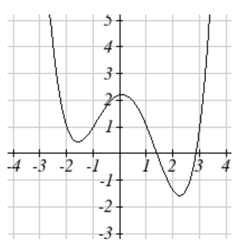
23. 

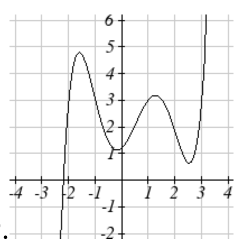
24. 

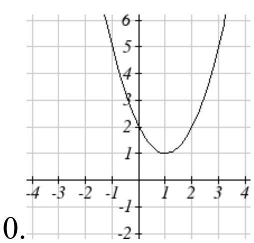
25. 

26. 

27. 

28. 

29. 

30. 

Find the vertical and horizontal intercepts of each function

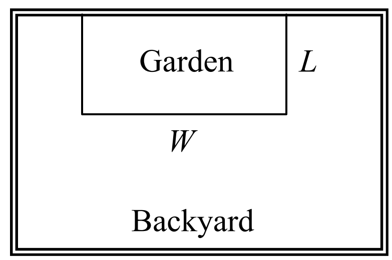
31. $f(t) = 2(t-1)(t+2)(t-3)$
32. $f(x) = 3(x+1)(x-4)(x+5)$
33. $g(n) = -2(3n-1)(2n+1)$
34. $k(u) = -3(4-n)(4n+3)$

Section 3.2 Quadratic Functions

In this section, we will explore the family of 2nd degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area of the fence if the sides of fencing perpendicular to the existing fence have length *L*.



In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, *W*, to represent the side of fencing parallel to the 4th side or backyard fence.

Since we know we only have 80 feet of fence available, we know that

$L + W + L = 80$, or more simply, $2L + W = 80$

This allows us to represent the width, *W*, in terms of L: $W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$A = LW = L(80 - 2L)$

$A(L) = 80L - 2L^2$

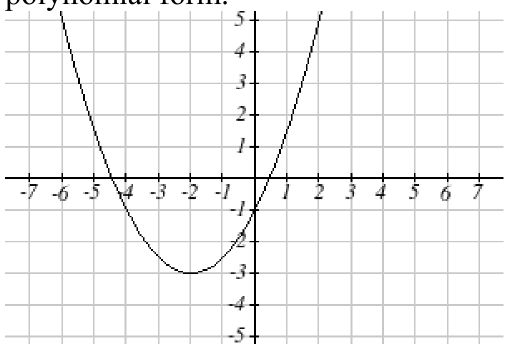
This formula represents the area of the fence in terms of the variable length *L*.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x) = x^2$.

Example 2

Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



We can see the graph is the basic quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a clear point such as (0,-1) we can solve for the stretch factor:

$$-1 = a(0+2)^2 - 3$$

$$2 = 4a$$

$$a = \frac{1}{2}$$

Written as a transformation, the equation for this formula is $g(x) = \frac{1}{2}(x+2)^2 - 3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$g(x) = \frac{1}{2}(x+2)^2 - 3$$

$$g(x) = \frac{1}{2}(x+2)(x+2) - 3$$

$$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3$$

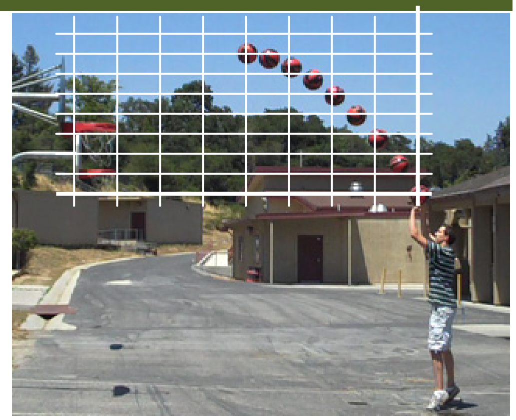
$$g(x) = \frac{1}{2}x^2 + 2x + 2 - 3$$

$$g(x) = \frac{1}{2}x^2 + 2x - 1$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Try it Now

1. A coordinate grid has been superimposed over the quadratic path of a basketball[1]. Find an equation for the path of the ball. Does he make the basket?



Forms of Quadratic Functions

The **standard form** of a quadratic is $f(x) = ax^2 + bx + c$

The **transformation form** of a quadratic is $f(x) = a(x-h)^2 + k$

The **vertex** of the quadratic is located at (h, k)

Because the vertex can also be seen in this format it is often called **vertex form** as well

In the previous example, we saw that it is possible to rewrite a quadratic in transformed form into standard form by expanding the formula. It would be useful to reverse this process, since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$$

$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b \text{, so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c \text{, so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

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[1] From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY

Finding Vertex of a Quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$$h = -\frac{b}{2a}, \quad k = f\left(\frac{-b}{2a}\right) = f(h)$$

Example 3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into transformation form (vertex form).

The horizontal component of the vertex will be at $h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$

The vertical component of the vertex will be at $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}$

Rewriting into transformation form, the stretch factor will be the same as the a in the original quadratic. Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Try it Now

2. Given the equation $g(x) = 13 + x^2 - 6x$ write the equation in Standard Form and then in Transformation/Vertex form.

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 4

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L) = 80L - 2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so a = -2, b = 80, and c = 0.

Finding the vertex:

$$h = -\frac{80}{2(-2)} = 20, \quad k = A(20) = 80(20) - 2(20)^2 = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 5

A local newspaper currently has 84,000 subscribers, at a quarterly cost of \$30. Market research has suggested that if they raised the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the cost, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the cost per subscription times the number of subscribers. We can introduce variables, C for cost per subscription and S for the number subscribers, giving us the equation
Revenue = CS

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $S = 84{,}000$ and $C = 30$, and that if they raise the price to \$32 they would lose 5,000 subscribers, giving a second pair of values, $C = 32$ and $S = 79{,}000$. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S = mC + b$. The slope will be

$$m = \frac{79{,}000 - 84{,}000}{32 - 30} = \frac{-5{,}000}{2} = -2{,}500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$S = -2500C + b$	Plug in the point $S = 85{,}000$ and $C = 30$
$84{,}000 = -2500(30) + b$	Solve for b
$b = 159{,}000$	

This gives us the linear equation $S = -2{,}500C + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

$\text{Revenue} = CS$	Substituting the equation for S from above
$\text{Revenue} = C(-2{,}500C + 159{,}000)$	Expanding
$\text{Revenue} = -2{,}500C^2 + 159{,}000C$	

We now have a quadratic equation for revenue as a function of the subscription cost. To find the cost that will maximize revenue for the newspaper, we can find the vertex:

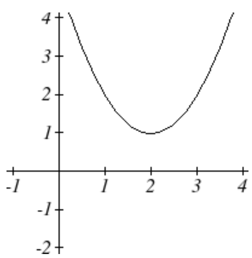
$$h = -\frac{159{,}000}{2(-2{,}500)} = 31.8$$

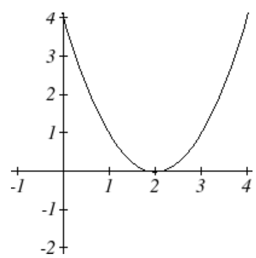
The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

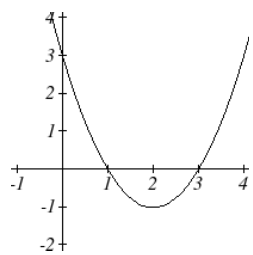
Maximum Revenue = $-2{,}500(31.8)^2 + 159{,}000(31.8) = \$2{,}528{,}100$

Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.


zero horizontal intercepts


one horizontal intercept


two horizontal intercepts

Example 6

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

We can find the vertical intercept by evaluating the function at an input of zero:
$f(0) = 3(0)^2 + 5(0) - 2 = -2$ Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero
$0 = 3x^2 + 5x - 2$

In this case, the quadratic can be factored, providing the simplest method for solution
$0 = (3x - 1)(x + 2)$

$0 = 3x - 1$, $x = \frac{1}{3}$ or $0 = x + 2$, $x = -2$ Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph.

Example 7

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Again we will solve for when the output will be zero

$0 = 2x^2 + 4x - 4$

Since the quadratic is not factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \qquad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$

$f(x) = 2(x+1)^2 - 6$

Now we can solve for when the output will be zero

$0 = 2(x+1)^2 - 6$

$6 = 2(x+1)^2$

$3 = (x+1)^2$

$x + 1 = \pm\sqrt{3}$

$x = -1 \pm \sqrt{3}$

The graph has horizontal intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$

Try it Now

3. In Try it Now problem 2 we found the standard & transformation form for the equation $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

Since this process is done commonly enough that sometimes people find it easier to solve the problem once in general then remember the formula for the result, rather than repeating the process. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solving for the horizontal intercepts using this general equation gives:

$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ start to solve for x by moving the constants to the other side

$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$ divide both sides by a

$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$ find a common denominator to combine fractions

$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ combine the fractions on the left side of the equation

$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ take the square root of both sides

$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$ subtract $b/2a$ from both sides

$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$ combining the fractions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic given in standard form, the **quadratic formula** gives the horizontal intercepts of the graph of the quadratic.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 8

A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $h(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground?

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, \quad k = h\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds

To find when the ball hits the ground, we need to determine when the height is zero – when *h(t)* = 0. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not evaluate to a whole number, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

Try it Now

4. For these two equations determine if the vertex will be a maximum value or a minimum value.
   a. $g(x) = -8x + x^2 + 7$
   b. $g(x) = -3(3 - x)^2 + 2$

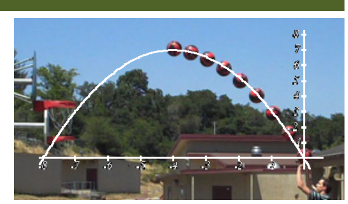
Important Topics of this Section

Quadratic functions
- Standard form
- Transformation form/Vertex form
- Vertex as a maximum / Vertex as a minimum

Short run behavior
- Vertex / Horizontal & Vertical intercepts

Quadratic formula

Try it Now Answers



1. The path passes through the origin with vertex at (-4, 7).
$h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, *h*(-7.5) would need to be about 4. $h(-7.5) \approx 1.64$; he doesn't make it.

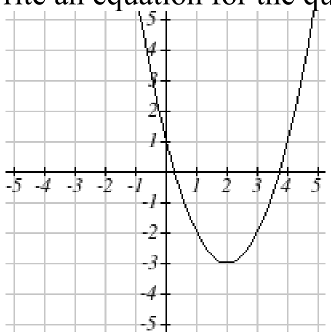
2. $g(x) = x^2 - 6x + 13$ in Standard form; $g(x) = (x-3)^2 + 4$ in Transformation form

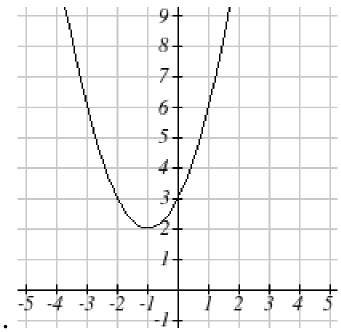
3. Vertical intercept at (0, 13), NO horizontal intercepts.

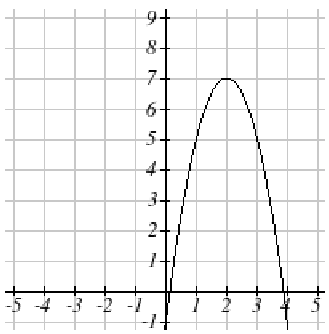
4. a. Vertex is a minimum value
   b. Vertex is a maximum value

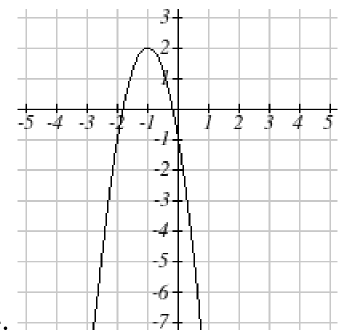
Section 3.2 Exercises

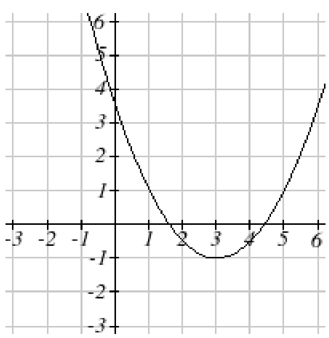
Write an equation for the quadratic graphed

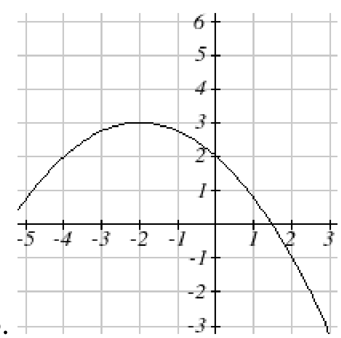
1. 

2. 

3. 

4. 

5. 

6. 

For each of the follow quadratics, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

7. $y(x) = 2x^2 + 10x + 12$

8. $z(p) = 3x^2 + 6x - 9$

9. $f(x) = 2x^2 - 10x + 4$

10. $g(x) = -2x^2 - 14x + 12$

11. $h(t) = -4t^2 + 6t - 1$

12. $k(t) = 2x^2 + 4x - 15$

Rewrite the quadratic into vertex form

13. $f(x) = x^2 - 12x + 32$

14. $g(x) = x^2 + 2x - 3$

15. $h(x) = 2x^2 + 8x - 10$

16. $k(x) = 3x^2 - 6x - 9$

17. Find the values of *b* and *c* so $f(x) = -8x^2 + bx + c$ has vertex $(2, -7)$
18. Find the values of *b* and *c* so $f(x) = 6x^2 + bx + c$ has vertex $(7, -9)$

Write an equation for a quadratic with the given features
19. *x*-intercepts (-3, 0) and (1, 0), and *y* intercept (0, 2)
20. *x*-intercepts (2, 0) and (-5, 0), and *y* intercept (0, 3)
21. *x*-intercepts (2, 0) and (5, 0), and *y* intercept (0, 6)
22. *x*-intercepts (1, 0) and (3, 0), and *y* intercept (0, 4)
23. Vertex at (4, 0), and *y* intercept (0, -4)
24. Vertex at (5, 6), and *y* intercept (0, -1)
25. Vertex at (-3, 2), and passing through (3, -2)
26. Vertex at (1, -3), and passing through (-2, 3)

27. A rocket is launched in the air. Its height, in meters above sea level, as a function of time is given by $h(t) = -4.9t^2 + 229t + 234$.
    a. From what height was the rocket launched?
    b. How high above sea level does the rocket get at its peak?
    c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

28. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time is given by $h(t) = -4.9t^2 + 24t + 8$.
    a. From what height was the ball thrown?
    b. How high above ground does the ball get at its peak?
    c. When does the ball hit the ground?

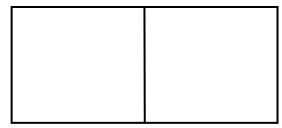
29. The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where *x* is the horizontal distance in feet from the point at which the ball is thrown.
    a. How high is the ball when it was thrown?
    b. What is the maximum height of the ball?
    c. How far from the thrower does the ball strike the ground?

30. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where *x* is the horizontal distance in feet from the point at which the javelin is thrown.
    a. How high is the javelin when it was thrown?
    b. What is the maximum height of the javelin?
    c. How far from the thrower does the javelin strike the ground?

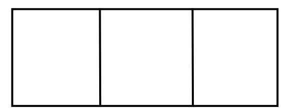
31. A box with a square base and no top is to be made from a square piece of cardboard by cutting 6 in. squares from each corner and folding up the sides. The box is to hold 1000 in^3. How big a piece of cardboard is needed?

32. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4 in. squares from each corner and folding up the sides. The box is to hold 2700 in^3. How big a piece of cardboard is needed?

33. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?



34. A farmer wishes to enclose three pens with fencing, as shown. If the farmer has 700 feet of fencing to work with, what dimensions will maximize the area enclosed?



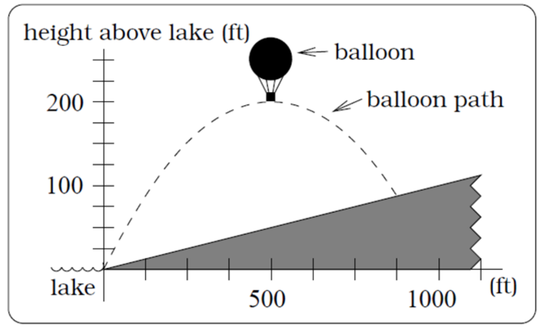
35. You have a wire that is 56 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a square. The other piece will be bent into the shape of a circle. Let A represent the total area of the square and the circle. What is the circumference of the circle when A is a minimum?

36. You have a wire that is 71 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a right triangle with base equal to height. The other piece will be bent into the shape of a circle. Let A represent the total area of the triangle and the circle. What is the circumference of the circle when A is a minimum?

37. A soccer stadium holds 62000 spectators. With a ticket price of \$11 the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

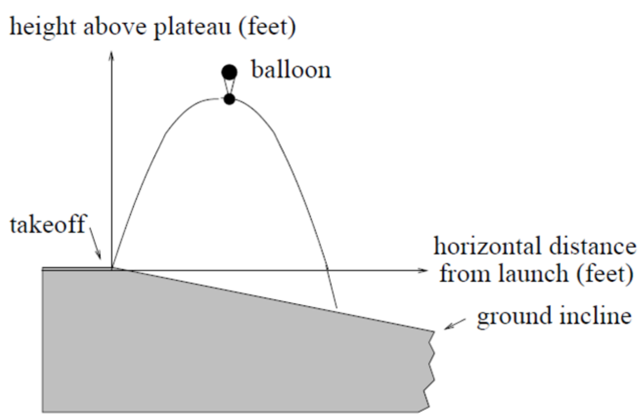
38. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

39. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of $f(x) = -\frac{2}{2500}x^2 + 45x$. The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

40. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

Section 3.3 Graphs of Polynomial Functions

In the previous section we explored the short run behavior of quadratics, a special case of polynomials. In this section we will explore the short run behavior of polynomials in general.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for any degree polynomial.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4th degree polynomials are not simple enough to remember, and formulas do not exist for general higher degree polynomials. Consequently, we will limit ourselves to three cases:

1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2) The polynomial is given in factored form
3) Technology is used to determine the intercepts

Example 1

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

We can attempt to factor this polynomial to find solutions for *f(x)* = 0

$x^6 - 3x^4 + 2x^2 = 0$ Factoring out the greatest common factor

$x^2\left(x^4 - 3x^2 + 2\right) = 0$ Factoring the inside as a quadratic

$x^2\left(x^2 - 1\right)\left(x^2 - 2\right) = 0$ Then break apart to find solutions

$x^2 = 0$, $x = 0$ or $\left(x^2 - 1\right) = 0$, $x^2 = 1$, $x = \pm 1$ or $\left(x^2 - 2\right) = 0$, $x^2 = 2$, $x = \pm\sqrt{2}$

This gives us 5 horizontal intercepts.

Example 2

Find the vertical and horizontal intercepts of $g(t) = (t-2)^2(2t+3)$

The vertical intercept can be found by evaluating *g*(0).

$g(0) = (0-2)^2(2(0)+3) = 12$

The horizontal intercepts can be found by solving *g(t)* = 0

$(t-2)^2(2t+3)=0$ Since this is already factored, we can break it apart:

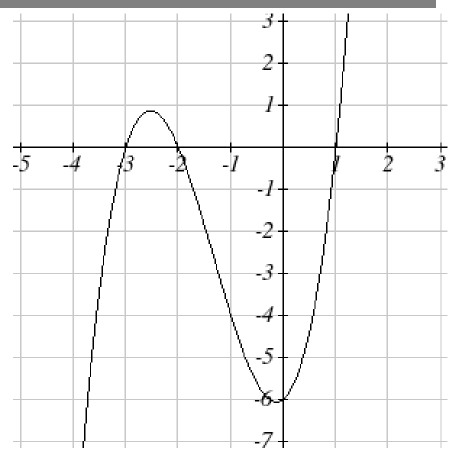
$(t-2)^2=0$ $(2t+3)=0$

$t-2=0$ or $t=\dfrac{-3}{2}$

$t=2$

Example 3

Find the horizontal intercepts of
$h(t)=t^3+4t^2+t-6$



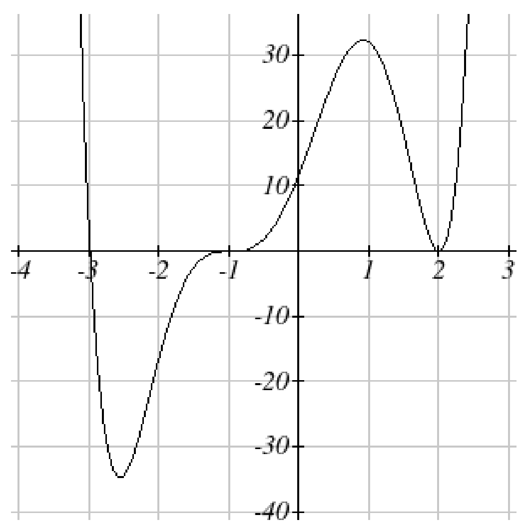
Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

Graphing this function, it appears there are horizontal intercepts at *x* = -3, -2, and 1

Try it Now

1. Find the vertical and horizontal intercepts of the function $f(t)=t^4-4t^2$

Graphical Behavior at Intercepts



If we graph the function $f(x)=(x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept *x* = -3, coming from the $(x+3)$ factor of the polynomial, the graph passes directly through the horizontal intercept. The factor is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero is formed from a single factor of the function.

At the horizontal intercept *x* = 2, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off of the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero.

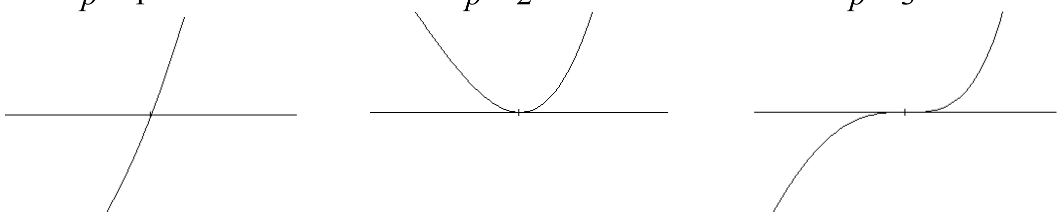
At the horizontal intercept $x = -1$, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit x^3 has. We call this a triple zero.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Graphical Behavior of Polynomials at Horizontal Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

$p = 1$ $\quad$ $p = 2$ $\quad$ $p = 3$



Single zero $\quad$ Double zero $\quad$ Triple zero

For higher even powers 4,6,8 etc… the graph will still bounce off of the graph but the graph will appear flatter with increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc… the graph will still pass through the graph but the graph will appear flatter with increasing odd power as it approaches and leaves the axis.

Example 4

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$

This graph has two horizontal intercepts. At $x = -3$, the factor is squared, indicating the graph will bounce at this horizontal intercept. At $x = 5$, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:
As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

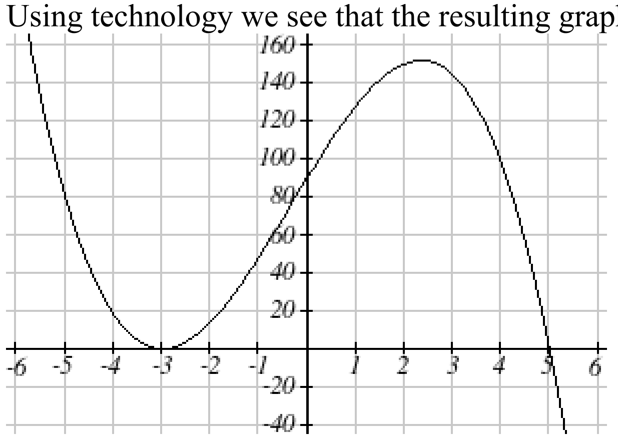
At (-3, 0) the graph bounces off of the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept

Somewhere after this point the graph must turn back down / or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0)

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we see that the resulting graph will look like:



Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 5

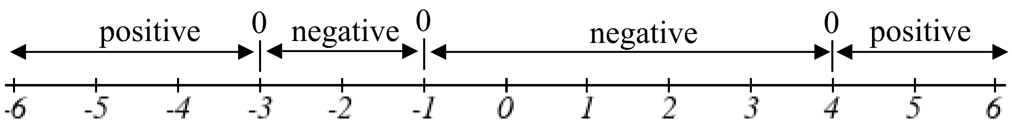
Solve $(x+3)(x+1)^2(x-4) > 0$

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4) = 0$, which has solutions at x = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	$f($ test value$)$	>0 or <0?
x < -3	-4	72	> 0
-3 < x < -1	-2	-6	< 0
-1 < x .< 4	0	-12	< 0
x > 4	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when x < -3 or x > 4, or in interval notation, $(-\infty,-3)\cup(4,\infty)$

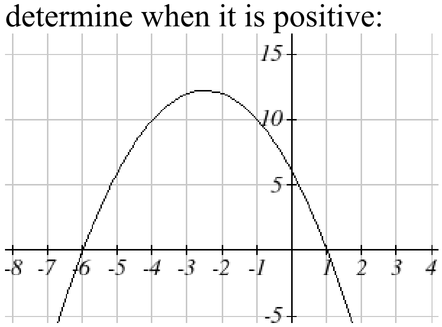
We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example

Example 6

Find the domain of the function $v(t) = \sqrt{6-5t-t^2}$

A square root only is defined when the quantity we are taking the square root of is zero or greater. Thus, the domain of this function will be when $6-5t-t^2 \ge 0$.

Again we start by solving the equality $6-5t-t^2 = 0$. While we could use the quadratic formula, this equation factors nicely to $(6+t)(1-t) = 0$, giving horizontal intercepts t = 1 and t = -6. Sketching a graph of this quadratic will allow us to determine when it is positive:



From the graph we can see this function is positive for inputs between the intercepts. So $6-5t-t^2 \ge 0$ for $-6 \le t \le 1$, and this will be the domain of the *v(t)* function.

Try it Now

2. Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Writing Equations using Intercepts

Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form an equation that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

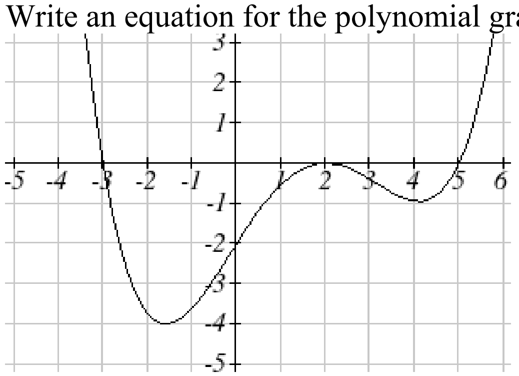
If a polynomial has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form

$$f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$$

where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor *a* can be determined given a value of the function other than the horizontal intercept.

Example 7

Write an equation for the polynomial graphed here



This graph has three horizontal intercepts: x = -3, 2, and 5. At x = -3 and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At x = 2 the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree or quadratic. Together, this gives us:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

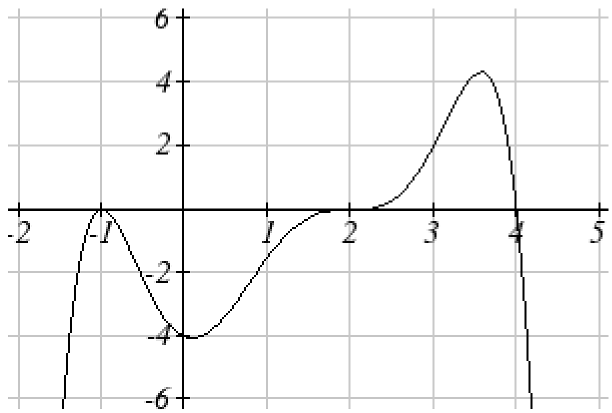
To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for *a*

$$-2 = a(0+3)(0-2)^2(0-5)$$
$$-2 = -60a$$
$$a = \frac{1}{30}$$

The graphed polynomial would have equation $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$

Try it Now

3. Given the graph, determine and write the equation for the graph in factored form.



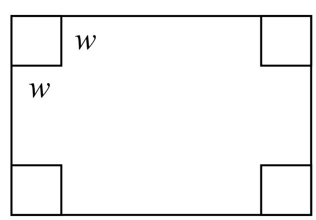
Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 8

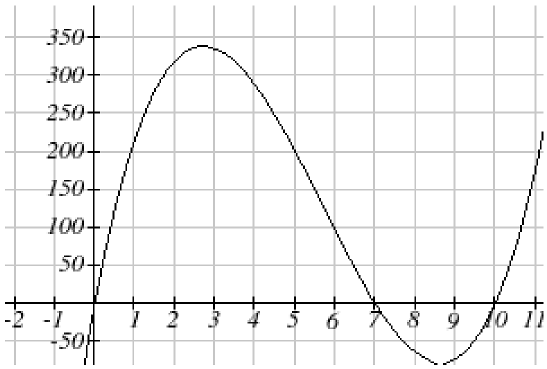
An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, *w*.

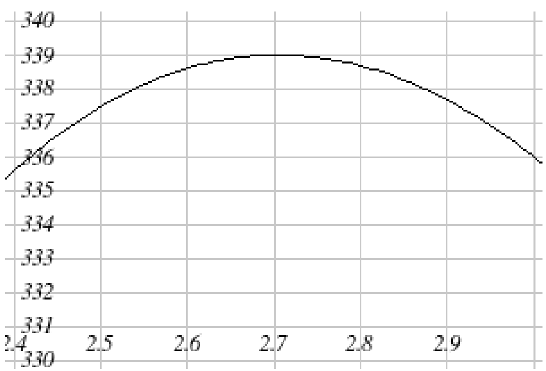


Notice that after a square is cut out from each end, it leaves (14-2*w*) cm by (20-2*w*) cm for the base of the box, and the box will be *w* cm tall. This gives the volume:
$V(w) = (14 - 2w)(20 - 2w)w = 280w - 68w^2 + 4w^3$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for *w* – values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use features of our technology if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

Try it Now

4. Use technology to find the Maximum and Minimum values on the interval [-1, 4] of the equation $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Important Topics of this Section

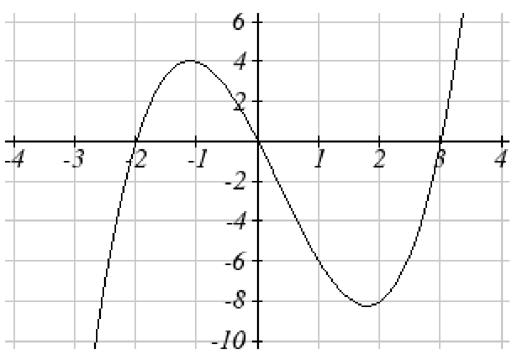
Short Run Behavior
- Intercepts (Horizontal & Vertical)

Methods to find Horizontal intercepts
- Factoring Methods
- Factored Forms
- Technology

Graphical Behavior at intercepts
Single, Double and Triple zeros (or power 1,2 & 3 behaviors)
Solving polynomial inequalities using test values & graphing techniques
Writing equations using intercepts
Estimating extrema

Try it Now Answers

1. Vertical intercept (0, 0) Horizontal intercepts (0, 0), (-2, 0), (2, 0)
2. Vertical intercept (0, 0) Horizontal intercepts (-2, 0), (0, 0), (3, 0)
   The function is negative from ($-\infty$, -2) and (0, 3)
   The function is positive from (-2, 0) and (3,∞)
   The leading term is x^3 so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$



3. $f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$
4. Approximately, (0, -6.5) minimum and approximately (3.5, 7) maximum.

Section 3.3 Exercises

Find the *C* and *t* intercepts of each function

1. $C(t)=2(t-4)(t+1)(t-6)$
2. $C(t)=3(t+2)(t-3)(t+5)$
3. $C(t)=4t(t-2)^2(t+1)$
4. $C(t)=2t(t-3)(t+1)^2$
5. $C(t)=2t^4-8t^3+6t^2$
6. $C(t)=4t^4+12t^3-40t^2$

Use your calculator or other graphing technology to solve graphically for the zeros of the function

7. $f(x)=x^3-7x^2+4x+30$
8. $g(x)=x^3-6x^2+x+28$

Find the long run behavior of each function as $t\to\infty$ and $t\to-\infty$

9. $h(t)=3(t-5)^3(t-3)^3(t-2)$
10. $k(t)=2(t-3)^2(t+1)^3(t+2)$
11. $p(t)=-2t(t-1)(3-t)^2$
12. $q(t)=-4t(2-t)(t+1)^3$

Sketch a graph of each equation

13. $f(x)=(x+3)^2(x-2)$
14. $g(x)=(x+4)(x-1)^2$
15. $h(x)=(x-1)^3(x+3)^2$
16. $k(x)=(x-3)^3(x-2)^2$
17. $m(x)=-2x(x-1)(x+3)$
18. $n(x)=-3x(x+2)(x-4)$

Solve each inequality

19. $(x-3)(x-2)^2>0$
20. $(x-5)(x+1)^2>0$
21. $(x-1)(x+2)(x-3)<0$
22. $(x-4)(x+3)(x+6)<0$

Find the domain of each function

23. $f(x)=\sqrt{-42+19x-2x^2}$
24. $g(x)=\sqrt{28-17x-3x^2}$
25. $h(x)=\sqrt{4-5x+x^2}$
26. $k(x)=\sqrt{2+7x+3x^2}$
27. $n(x)=\sqrt{(x-3)(x+2)^2}$
28. $m(x)=\sqrt{(x-1)^2(x+3)}$
29. $p(t)=\dfrac{1}{t^2+2t-8}$
30. $q(t)=\dfrac{4}{x^2-4x-5}$

Write an equation for a polynomial the given features

31. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. Vertical intercept at (0, -4)

32. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. Vertical intercept at (0, 6)

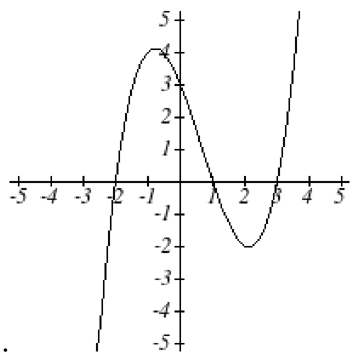
33. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. Vertical intercept at (0, 9)

34. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. Vertical intercept at (0, -3)

35. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point (2, 15)

36. Degree 5. Single zero at $x = -2$ and $x = 3$, and triple zero at $x = 1$. Passes through the point (2, 4)

Write an equation for the polynomial graphed

37.


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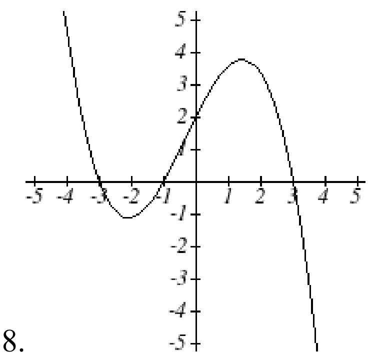

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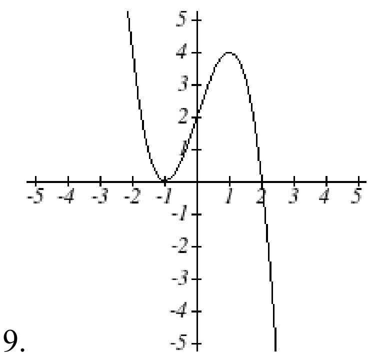

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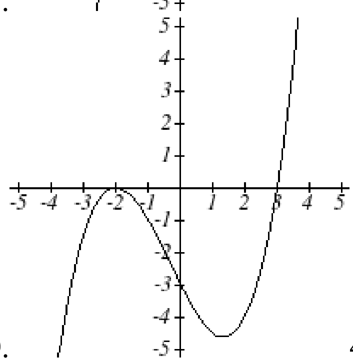

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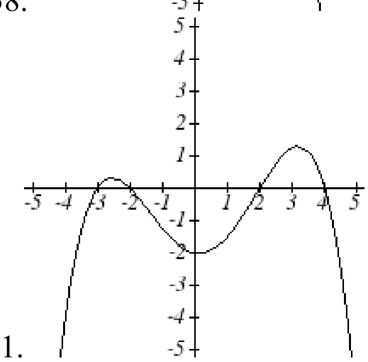

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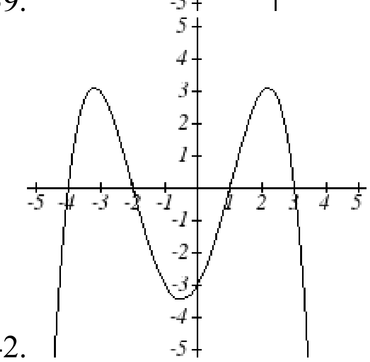

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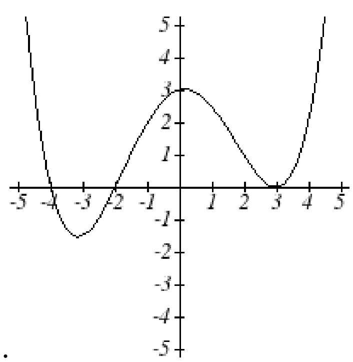

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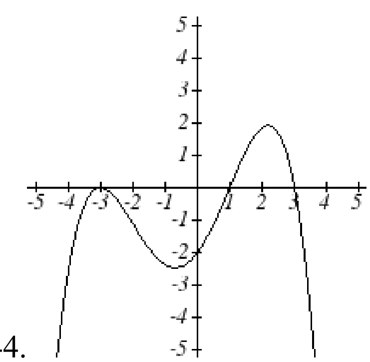
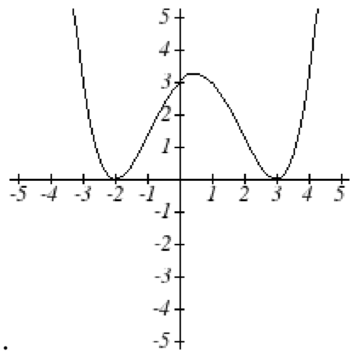
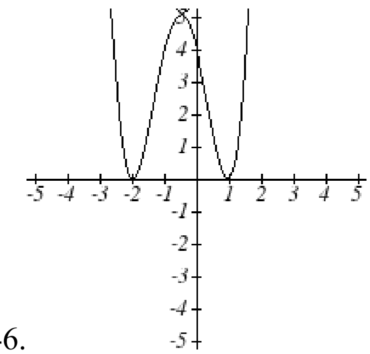
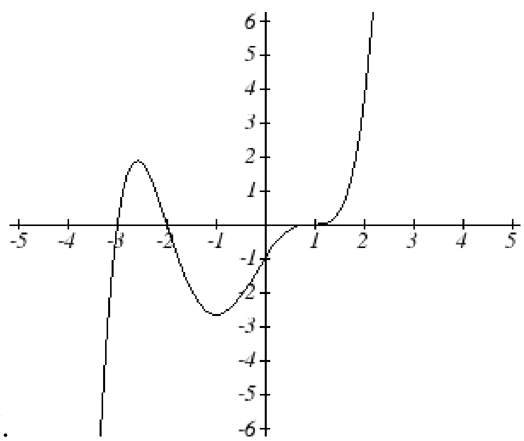
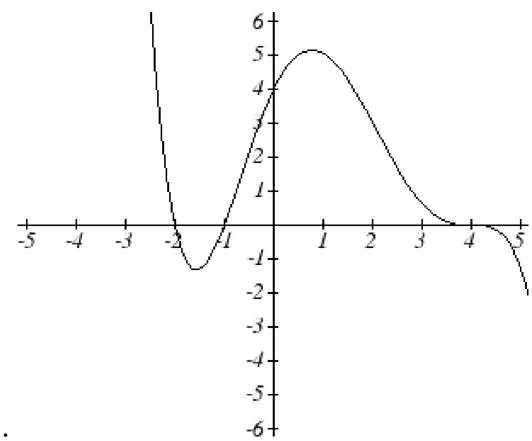


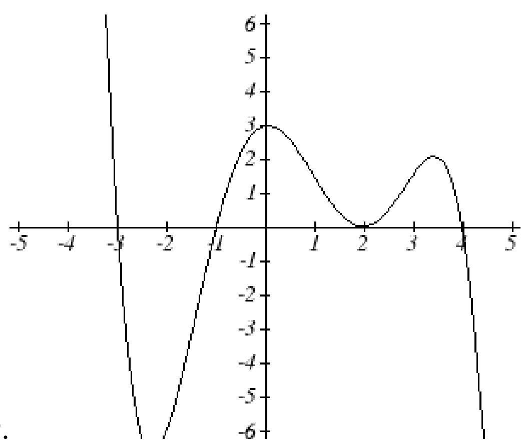
Write an equation for the polynomial graphed

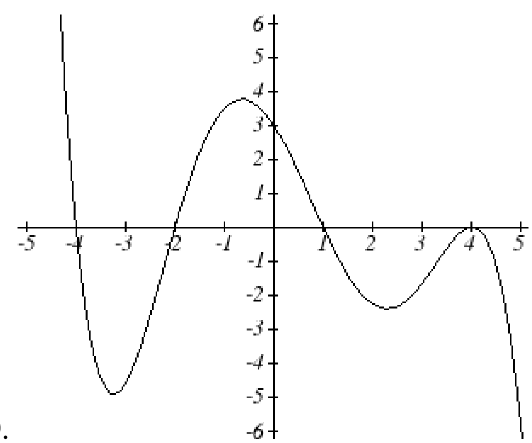
45. 

46. 

47. 

48. 

49. 

50. 

51. A rectangle is inscribed with its base on the *x* axis and its upper corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

52. A rectangle is inscribed with its base on the *x* axis and its upper corners on the curve $y = 16 - x^4$. What are the dimensions of such a rectangle with the greatest possible area?

Section 3.4 Rational Functions

In the last few sections, we have built polynomials based on the positive whole number power functions. In this section we explore the functions based on power functions with negative integer powers, the rational functions.

Example 1

You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then $vt = \text{distance}$. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us the function we desired:

$$t(v) = \frac{100}{v} = 100v^{-1}$$

While this type of relationship can be written using the negative exponent, it is more common to see it written as a fraction.

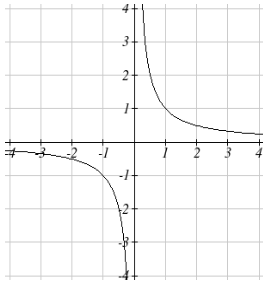
This particular example is one of an **inversely proportional** relationship – where one quantity is a constant divided by the other quantity. $f(x) = \frac{1}{x}$

Notice that this is a transformation of the reciprocal toolkit function.

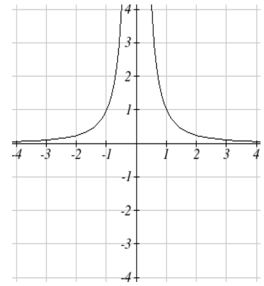
Several natural phenomena, such as gravitational force and volume of sound, behave in a manner **inversely proportional to the square** of the second quantity. For example, the volume, V, of a sound heard at a distance d from the source would be related by $V = \frac{k}{d^2}$ for some constant value k.

These functions are transformations of the reciprocal squared toolkit function $f(x) = \frac{1}{x^2}$

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. These graphs have several important features.



$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. As you well know, dividing by zero is not allowed and therefore zero is not in the Domain, and so the function is undefined at an input of zero.

Short run behavior:
As the input becomes very small or as the input values approach zero from the left side, the function values become very large in a negative direction, or approach negative infinity.
We write: as $x \to 0^-$, $f(x) \to -\infty$.

As we approach 0 from the right side, the input values are still very small, but the function values become very large or approach positive infinity.
We write: as $x \to 0^+$ $f(x) \to \infty$.

This behavior creates a **vertical asymptote**. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line $x = 0$ as the input becomes close to zero.

Long run behavior:
As the values of x approach infinity, the function values approach 0.
As the values of x approach negative infinity, the function values approach 0.
Symbolically, as $x \to \pm\infty$ $f(x) \to 0$

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive direction.

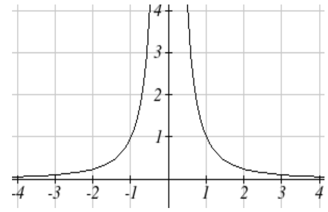
Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a. As $x \to a$, $f(x) \to \pm\infty$.

A **horizontal asymptote** of a graph is a horizontal line $f(x) = b$ where the graph approaches the line as the inputs get large. As $x \to \pm\infty$, $f(x) \to b$.

Try it Now:

1. Use symbolic notation to describe the long run behavior and short run behavior for the reciprocal squared function.



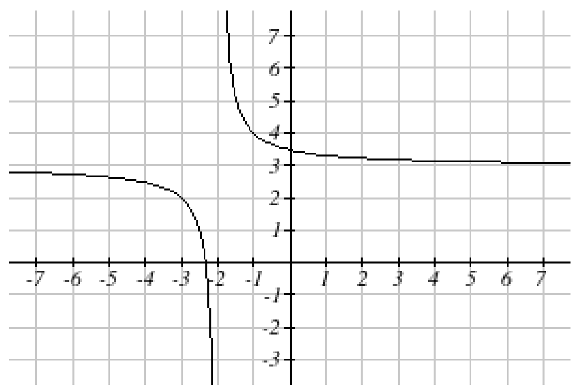
Example 2

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Transforming the graph left 2 and up 3 would result in the equation

$f(x) = \dfrac{1}{x+2} + 3$, or equivalently by giving the terms a common denominator,

$f(x) = \dfrac{3x+7}{x+2}$



Shifting the toolkit function would give us this graph. Notice that this equation is undefined at x = -2, and the graph also is showing a vertical asymptote at x = -2.

As $x \to -2^-$, $f(x) \to -\infty$, and as $x \to -2^+$, $f(x) \to \infty$

As the inputs grow large, the graph appears to be leveling off at $f(x) = 3$, indicating a horizontal asymptote at $f(x) = 3$.

As $x \to \pm\infty$, $f(x) \to 3$.

Notice that horizontal and vertical asymptotes shifted along with the function.

Try it Now

2. Sketch the graph and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

In the previous example, we shifted the function in a way that resulted in a function of the form $f(x) = \dfrac{3x+7}{x+2}$. This is an example of a general rational function.

Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials, *p(x)* and *q(x)*.

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_0+a_1x+a_2x^2+\cdots+a_px^p}{b_0+b_1x+b_2x^2+\cdots+b_qx^q}$$

Example 3

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after *t* minutes.

Notice that the water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$water=100+10t$

$sugar=5+1t$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t}$$

Finding Asymptotes and Intercepts

Given a rational equation, as part of discovering the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts as we have in the past.

To find vertical asymptotes, we notice that the vertical asymptotes occurred when the denominator of the function was undefined. With few exceptions, a vertical asymptote will occur whenever the denominator is undefined.

Example 4

Find the vertical asymptotes of the function $k(x)=\frac{5+2x^2}{2-x-x^2}$

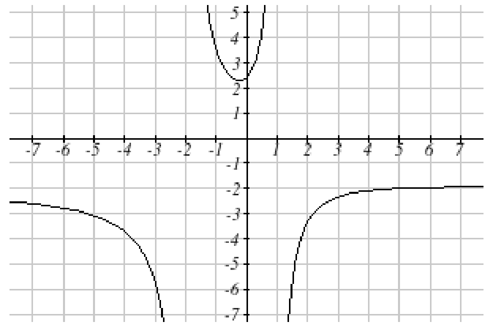
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$2-x-x^2=0$

$(2+x)(1-x)=0$

$x=-2,1$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule occurs when both the numerator and denominator of a rational function are zero.

Example 5

Find the vertical asymptotes of the function $k(x) = \dfrac{x-2}{x^2-4}$

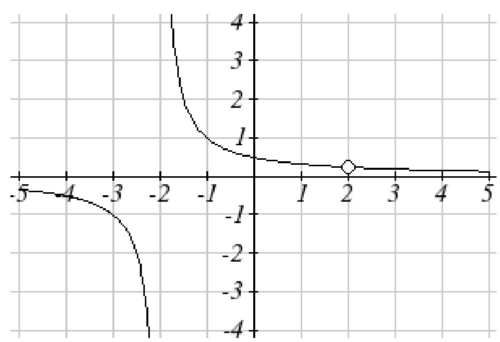
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$x^2 - 4 = 0$

$x^2 = 4$

$x = -2, 2$

However, the numerator of this function is also equal to zero when $x = 2$. Because of this, the function will still be undefined at 2, since $\dfrac{0}{0}$ is still undefined, but the graph will not have a vertical asymptote at $x = 2$.



The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole; a single point where the graph is not defined, indicated by an open circle.

Vertical Asymptotes and Holes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** will occur in a rational function if an input causes both the numerator and denominator to both be zero.

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

Case 1: The degree of the denominator > degree of the numerator

Example: $f(x)=\dfrac{3x+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x)=\dfrac{3x}{x^2}=\dfrac{3}{x}$. This tells us that as the inputs grow large, this function will behave similarly to the function $f(x)=\dfrac{3}{x}$. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at $f(x)=0$.
As $x \to \pm\infty$, $f(x) \to 0$

Case 2: The degree of the denominator < degree of the numerator

Example: $f(x)=\dfrac{3x^2+2}{x-5}$

In this case, the long run behavior is $f(x)=\dfrac{3x^2}{x}=3x$. This tells us that as the inputs grow large, this function will behave similarly to the function $f(x)=3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. Instead, the graph will approach the slanted line $f(x)=3x$.
As $x \to \pm\infty$, $f(x) \to \pm\infty$, respectively.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As another example if we had the function $f(x)=\dfrac{3x^5-x^2}{x+3}$ with long run behavior $f(x)=\dfrac{3x^5}{x}=3x^4$, the long run behavior of the graph would look similar to that of an even polynomial and as $x \to \pm\infty$, $f(x) \to \infty$.

Case 3: The degree of the denominator = degree of the numerator

Example: $f(x)=\dfrac{3x^2+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) = \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave the similarly to the function $f(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $f(x) = 3$.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.
Degree of denominator > degree of numerator: Horizontal asymptote at $f(x) = 0$
Degree of denominator < degree of numerator: No horizontal asymptote
Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 6

In the sugar concentration problem from earlier, we created the equation
$C(t) = \frac{5+t}{100+10t}$.
Find the horizontal asymptote and interpret it in context of the scenario.

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: As $x \to \pm\infty$, $f(x) \to \frac{1}{10}$. This function will have a horizontal asymptote at $f(x) = \frac{1}{10}$.

This tells us that as the input gets large, the output values will approach 1/10. In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.

Example 7

Find the horizontal and vertical asymptotes of the function
$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

The function will have vertical asymptotes when the denominator is zero causing the function to be undefined. The denominator will be zero at $x = 1$, -2, and 5, indicating vertical asymptotes at these values.

The numerator is degree 2, while the denominator is degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $f(x) = 0$.

Try it Now

3. Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero. It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 8

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

We can find the vertical intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$ This is equivalent to when the numerator is zero

$$0 = (x-2)(x+3)$$

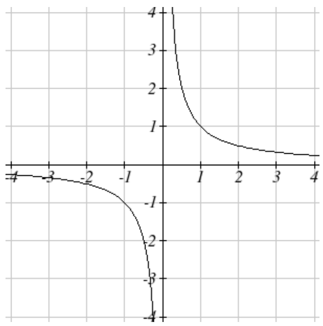
$$x = 2, -3$$

Try it Now

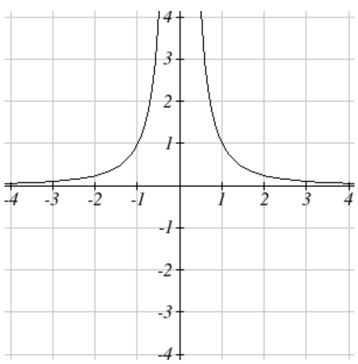
4. Given the reciprocal squared function that is shifted right 3 units and down 4 units. Write this as a rational function and find the horizontal and vertical intercepts and the horizontal and vertical asymptotes.

From the previous example, you probably noticed that the numerator of a rational function reveals the horizontal intercepts of the graph, while the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have powers. Happily, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

When factors of the denominator have power, the behavior at that intercept will mirror one of the two toolkit reciprocal functions.

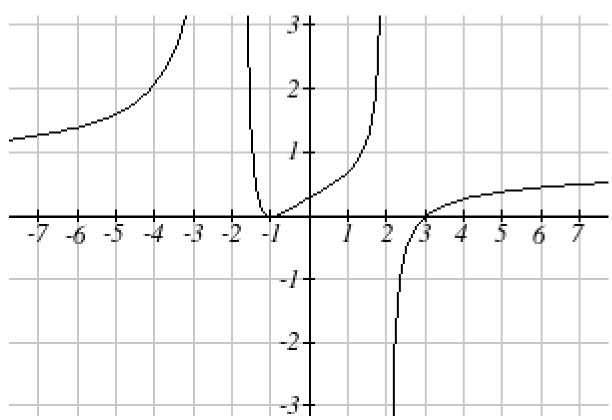


We get this behavior when the degree of the factor in the denominator is odd. The distinguishing characteristic is that on one side of the vertical asymptote the graph increases, and on the other side the graph decreases.



We get this behavior when the degree of the factor in the denominator is even. The distinguishing characteristic is that on both sides of the vertical asymptote the graph either increases or decreases.

For example, the graph of $f(x)=\dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown here.



At the horizontal intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces at the intercept, consistent with the quadratic nature of the factor.

At the horizontal intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we'd expect from a linear factor.

At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph increases on both sides of the asymptote, consistent with the behavior of the $\dfrac{1}{x^2}$ toolkit.

At the vertical asymptote $x = 2$ corresponding to the $(x-2)$ factor of the denominator, the graph increases on the left side of the asymptote and decreases as the inputs approach the asymptote from the right side, consistent with the behavior of the $\frac{1}{x}$ toolkit.

Example 9

Sketch a graph of $f(x) = \dfrac{(x+2)(x-3)}{(x+1)^2(x-2)}$

We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

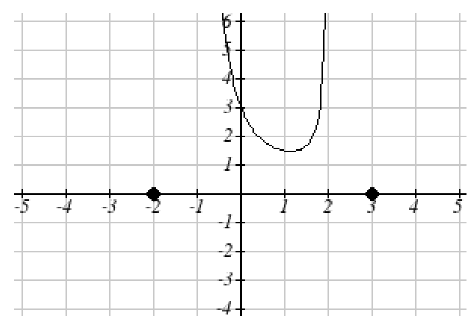
$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

Looking at when the numerator of the function is zero, we can determine the graph will have horizontal intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear, with the graph passing through the intercept.

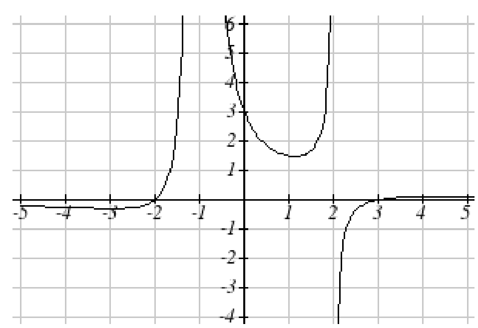
Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at $x = -1$ and $x = 2$.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no horizontal intercepts between the vertical asymptotes, and the vertical intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph.



Since the factor associated with the vertical asymptote at $x = -1$ was squared, we know the graph will have the same behavior on both sides of the asymptote. Since the graph increases as the inputs approach the asymptote on the right, the graph will increase as the inputs approach the asymptote on the left as well. For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote.



After passing through the horizontal intercepts, the graph will then level off towards an output of zero, as indicated by the horizontal asymptote.

Try it Now

5. Given the function $f(x)=\dfrac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe the behavior and sketch the function .

Since a rational function written in factored form will have a horizontal intercept where each factor of the numerator is equal to zero, we can form a numerator that will pass through a set of horizontal intercepts by introducing a corresponding set of factors. Likewise since the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will exhibit the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

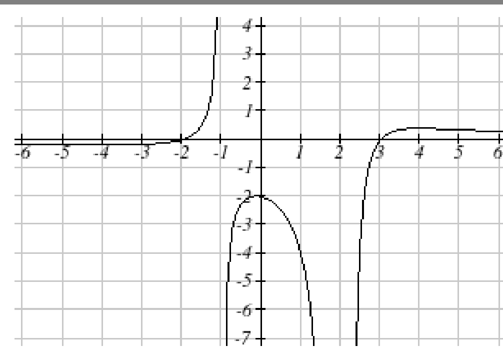
If a rational function has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, and vertical asymptotes at $x = v_1, v_2, \ldots, v_m$ then the function can be written in the form

$$f(x)=a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

Example 10

Write an equation for the rational function graphed here.



The graph appears to have horizontal intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors.

The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph increasing on one side and decreasing on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph decreasing on both sides of the asymptote.

Utilizing this information indicates an equation of the form
$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$

To find the stretch factor, we can use another clear point on the graph, such as the vertical intercept (0,-2)
$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$
$-2 = a\frac{-6}{4}$
$a = \frac{-8}{-6} = \frac{4}{3}$

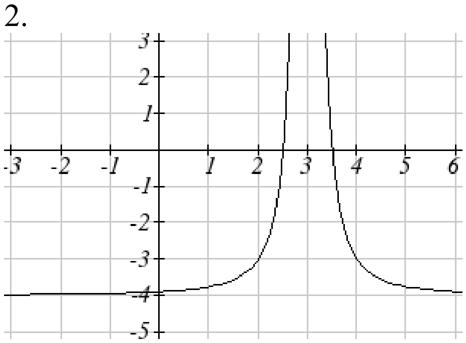
This gives us a final equation of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$

Important Topics of this Section

Inversely proportional; Reciprocal toolkit function
Inversely proportional to the square; Reciprocal squared toolkit function
Horizontal Asymptotes
Vertical Asymptotes
Rational Functions
- Finding intercepts, asymptotes, and holes.
- Given equation sketch the graph
- Identifying the function from a graph

Try it Now Answers

1. Long run behavior, as $x \to \pm\infty$, $f(x) \to 0$
   Short run behavior, as $x \to 0$, $f(x) \to \infty$ (there are no horizontal or vertical intercepts)

2.


The function and the asymptotes are shifted 3 units right and 4 units down.
As $x \to 3$, $f(x) \to \infty$ and as $x \to \pm\infty$, $f(x) \to -4$

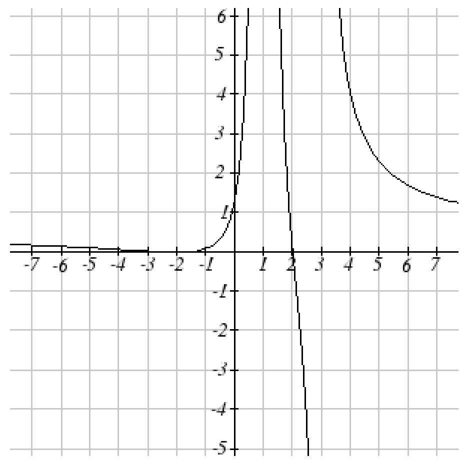
3. Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$

4. For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Since the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$. $f(x) = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero to find the vertical asymptote at $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We set the numerator equal to 0 and find the horizontal intercepts are at (2.5,0) and (3.5,0), then we evaluate at 0 and the vertical intercept is at $\left(0, \frac{-35}{9}\right)$

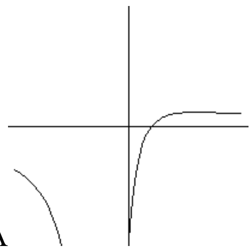
5.
Horizontal asymptote at $y = 1/2$.
Vertical asymptotes are at $x = 1$, and $x = 3$.
Vertical intercept at (0, 4/3),
Horizontal intercepts (2, 0) and (-2, 0)
(-2, 0) is a double zero and the graph bounces off the axis at this point.
(2, 0) is a single zero and crosses the axis at this point.



Section 3.4 Exercises

Match each equation form with one of the graphs

1. $f(x)=\dfrac{x-A}{x-B}$ 2. $g(x)=\dfrac{(x-A)^2}{x-B}$ 3. $h(x)=\dfrac{x-A}{(x-B)^2}$ 4. $k(x)=\dfrac{(x-A)^2}{(x-B)^2}$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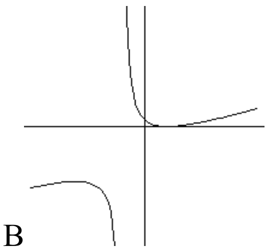
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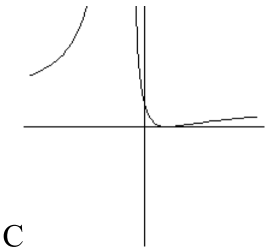
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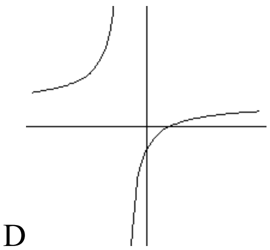

D

For each function, find the x intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote. Use that information to sketch a graph.

5. $p(x)=\dfrac{2x-3}{x+4}$

6. $q(x)=\dfrac{x-5}{3x-1}$

7. $s(x)=\dfrac{4}{(x-2)^2}$

8. $r(x)=\dfrac{5}{(x+1)^2}$

9. $f(x)=\dfrac{3x^2-14x-5}{3x^2+8x-16}$

10. $g(x)=\dfrac{2x^2+7x-15}{3x^2-14+15}$

11. $a(x)=\dfrac{x^2+2x-3}{x^2-1}$

12. $b(x)=\dfrac{x^2-x-6}{x^2-4}$

13. $h(x)=\dfrac{2x^2+x-1}{x-4}$

14. $k(x)=\dfrac{2x^2-3x-20}{x-5}$

15. $n(x)=\dfrac{3x^2+4x-4}{x^3-4x^2}$

16. $m(x)=\dfrac{5-x}{2x^2+7x+3}$

17. $w(x)=\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

18. $z(x)=\dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

Write an equation for a rational function with the given characteristics

19. Vertical asymptotes at $x = 5$ and $x = -5$
   x intercepts at $(2,0)$ and $(-1,0)$ y intercept at $(0,4)$

20. Vertical asymptotes at $x = -4$ and $x = -1$
   x intercepts at $(1,0)$ and $(5,0)$ y intercept at $(0,7)$

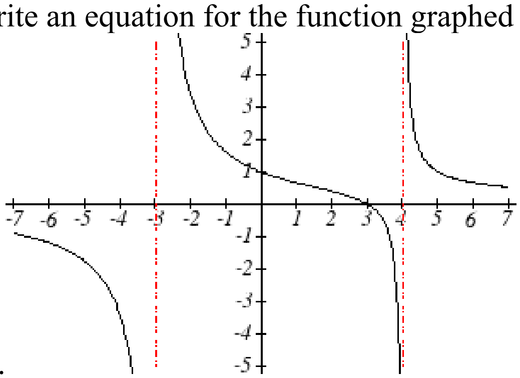
21. Vertical asymptotes at $x = -4$ and $x = -5$
   x intercepts at $(4,0)$ and $(-6,0)$ Horizontal asymptote at $y = 7$

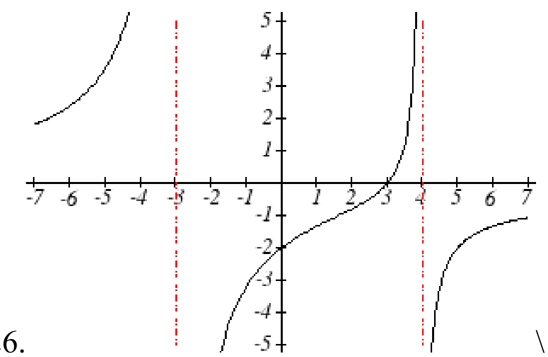
22. Vertical asymptotes at $x = -3$ and $x = 6$
   x intercepts at $(-2,0)$ and $(1,0)$ Horizontal asymptote at $y = -2$

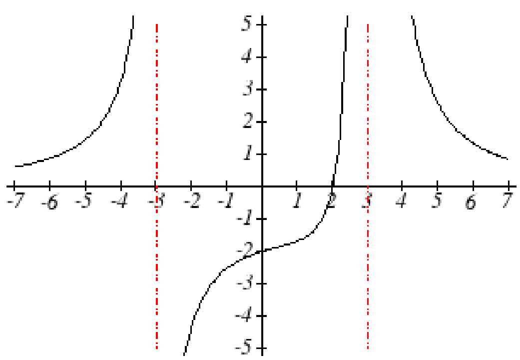
23. Vertical asymptote at $x = -1$
   Double zero at $x = 2$ y intercept at $(0,2)$

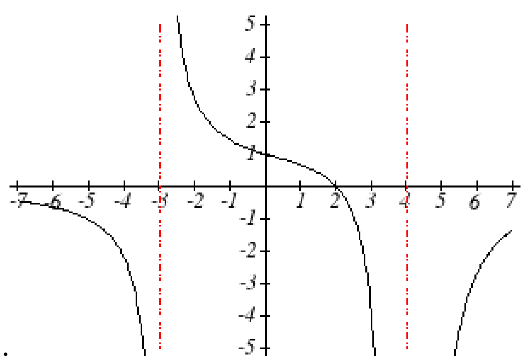
24. Vertical asymptote at $x = 3$
   Double zero at $x = 1$ y intercept at $(0,4)$

Write an equation for the function graphed

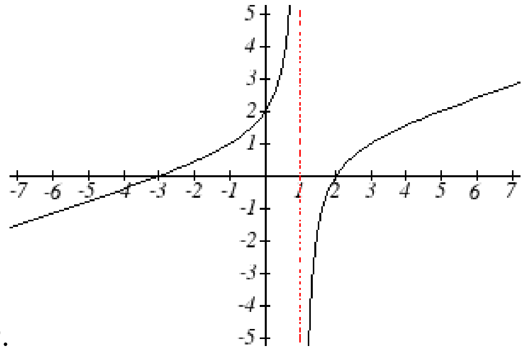
25. 

26. 

27. 

28. 

Write an equation for the function graphed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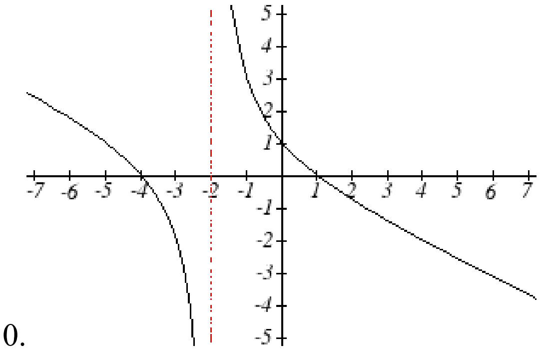
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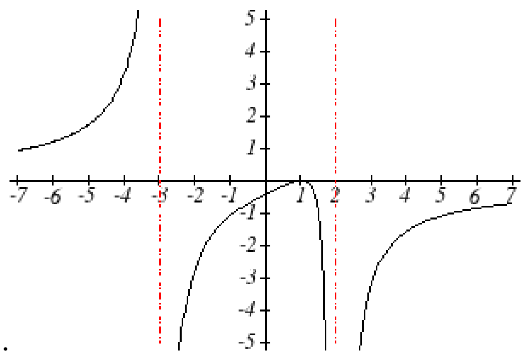
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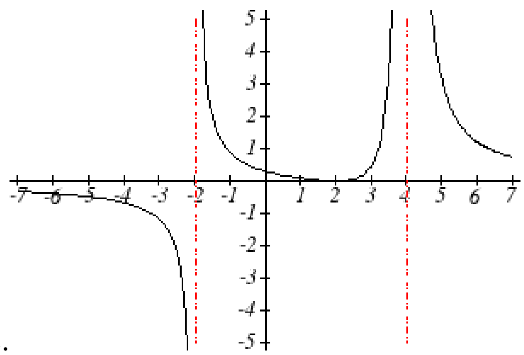
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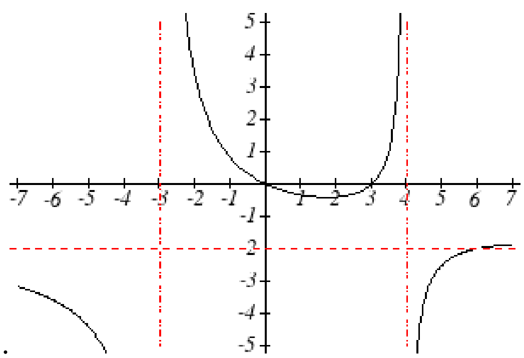
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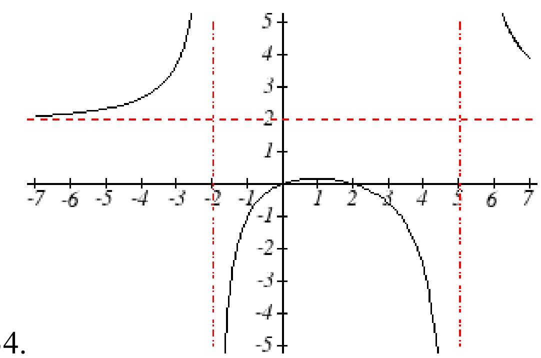
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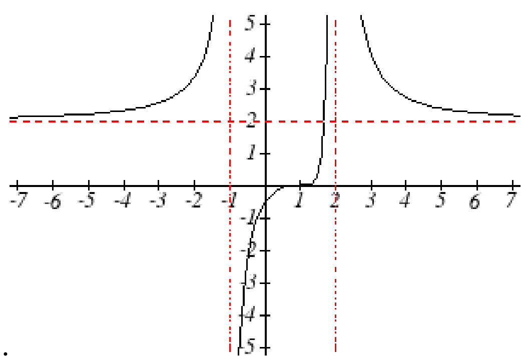
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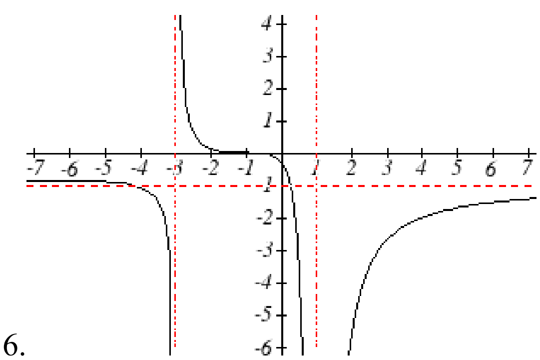
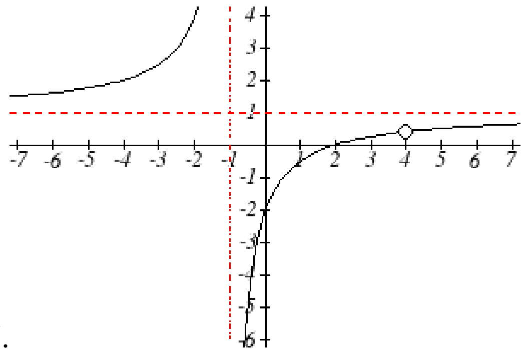
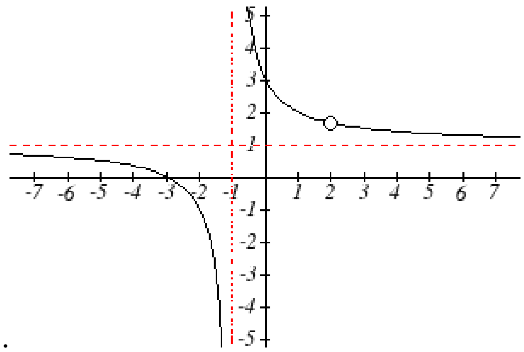

36.

Write an equation for the function graphed

37. 

38. 

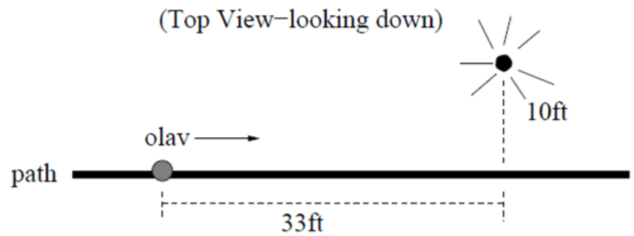
39. A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water.
    a. Write an equation for the concentration in the beaker after adding *n* mL of water
    b. Find the concentration if 10 mL of water is added
    c. How many mL of water must be added to obtain a 4% solution?
    d. What is the behavior as $n \to \infty$, and what is the physical significance of this?

40. A scientist has a beaker containing 30 mL of a solution containing 3 grams of potassium hydroxide. To this, she mixes a solution containing 8 milligrams per mL of potassium hydroxide.
    a. Write an equation for the concentration in the tank after adding *n* mL of the second solution.
    b. Find the concentration if 10 mL of the second solution is added
    c. How many mL of water must be added to obtain a 50 mg/mL solution?
    d. What is the behavior as $n \to \infty$, and what is the physical significance of this?

41. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4. [UW]
    a. Give a rational model of form $m(x) = \dfrac{ax+b}{cx+d}$ relating the meter reading $m(x)$ to how many feet x Oscar has gone into the room.
    b. How far must he go for the meter to reach 10? 100?
    c. Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

42. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score, $p(n)$, is a function of the number of hours, n, that you study in the form $p(n) = \frac{an+b}{cn+d}$. If you want a score of 80%, how long do you need to study? [UW]

43. A street light is 10 feet North of a straight bike path that runs East-West. Olav is bicycling down the path at a rate of 15 MPH. At noon, Olav is 33 feet West of the point on the bike path closest to the street light. (See the picture).



The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source. [UW]

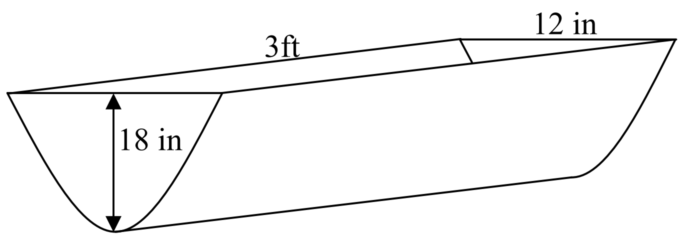
a. From 20 feet away, the street light has an intensity of 1 candle. What is k?
b. Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.
c. When will the light on Olav have maximum intensity?
d. When will the intensity of the light be 2 candles?

Section 3.5 Inverses and Radical Functions

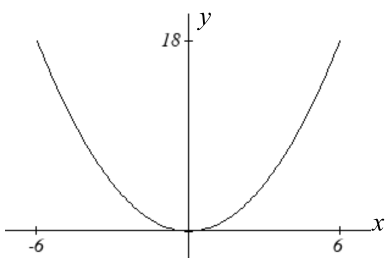
In this section, we will explore the inverses of polynomial and rational functions, and in particular the radical functions that arise from finding the inverses of quadratic functions.

Example 1

A parabolic trough water runoff collector is built as shown below. Find the surface area of the water in the trough as a function of the depth of the water.



Since it will be helpful to have an equation for the parabolic cross sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola.



From this we find an equation for the parabolic shape. Since we placed the origin at the vertex of the parabola, we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6,18), from which we can solve for the stretch factor a:

$18 = a6^2$

$a = \frac{18}{36} = \frac{1}{2}$

Our parabolic cross section has equation $y(x) = \frac{1}{2}x^2$

Since we are interested in the surface area of the water, we are interested in determining the width at the top of the water as a function of the water depth. This is the inverse of the function we just determined. However notice that the original function is not one-to-one, and indeed given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$y = \frac{1}{2}x^2$$

$$2y = x^2$$

$$x = \pm\sqrt{2y}$$

This is not a function as written. Since we are limiting ourselves to positive x values, we eliminate the negative solution, giving us the inverse function we're looking for

$$x(y) = \sqrt{2y}$$

Since x measures from the center out, the entire width of the water at the top will be $2x$. Since the trough is 3 feet (36 inches) long, the surface area will then be $36(2x)$, or in terms of y:

$$Area = 72x = 72\sqrt{2y}$$

The previous example illustrated two important things:

1) When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2) The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**.

Example 2

Find the inverse of $f(x) = (x-2)^2 - 3 = x^2 - 4x + 1$

From the transformation form of the equation, we can see the vertex is at (2,-3), and that it behaves like a basic quadratic. Since the graph will be decreasing on one side of the vertex, and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we start by writing the function in standard polynomial form, replacing the *f(x)* with a simple variable y. Since this is a quadratic equation, we know that to solve it for x we will want to arrange the equation so that it is equal to zero, which we can do by subtracting y from both sides of the equation.

$$y = x^2 - 4x + 1$$

$$0 = x^2 - 4x + 1 - y$$

In this format there is no easy way to algebraically put x on one side & everything else on the other, but we can recall that given a basic quadratic in standard form $f(x) = ax^2 + bx + c$ we can solve for x by using the quadratic formula

$x = \frac{-(b) \pm \sqrt{(b)^2 - 4(a)(c)}}{2a}$. We solve apply this to our equation $0 = x^2 - 4x + 1 - y$ by using $a = 1$, $b = -4$, and $c = (1 - y)$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1-y)}}{2} = 2 \pm \frac{\sqrt{12 + 4y}}{2}$$

Of course, as written this is not a function. Since we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case:

$$x = f^{-1}(y) = 2 + \frac{\sqrt{12 + 4y}}{2}$$

Try it Now

1. Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$

While it is not possible to find an inverse of most polynomial functions, some other basic polynomials are invertible.

Example 3

Find the inverse of the function $f(x) = 5x^3 + 1$

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for *x*

$$y = 5x^3 + 1$$

$$y - 1 = 5x^3$$

$$\frac{y - 1}{5} = x^3$$

$$x = f^{-1}(y) = \sqrt[3]{\frac{y - 1}{5}}$$

Notice that this inverse is also a transformation of a power function with a fractional power, $x^{1/3}$.

Try it Now

2. Which toolkit functions have inverse functions without restricting their domain?

Besides being important as an inverse function, radical functions are common in important physical models.

Example 4

The velocity, v in feet per second, of a car that slams on its brakes can be determined based on the length of skid marks that the tires leave on the ground. This relationship is given by

$$v(d) = \sqrt{2gfd}$$

In this formula, g represents acceleration due to gravity (32 ft/sec^2), d is the length of the skid marks in feet, and f is a constant representing the friction of the surface. A car lost control on wet asphalt, with a friction coefficient of 0.5, leaving 200 foot skid marks. How fast was the car travelling when it lost control?

Using the given values of $f = 0.5$ and $d = 200$, we can evaluate the given formula:
$v(200) = \sqrt{2(32)(0.5)(200)} = 80\, ft/\sec$, which is about 54.5 miles per hour.

Radical functions raise important question of domain when composed with more complicated functions.

Example 5

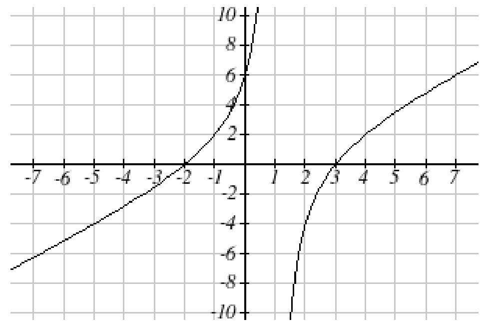
Find the domain of the function $f(x) = \sqrt{\dfrac{(x+2)(x-3)}{(x-1)}}$

Since a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\dfrac{(x+2)(x-3)}{(x-1)} \geq 0$. A rational function can change signs (change from positive to negative or vice versa) at horizontal intercepts and at vertical asymptotes. For this equation, the graph could change signs at x = -2, 1, and 3.

To determine on which intervals the rational expression is positive, we could evaluate the expression at test values, or sketch a graph. While both approaches work equally well, for this example we will use a graph.

This function has two horizontal intercepts, both of which exhibit linear behavior, where the graph will pass through the intercept. There is one vertical asymptote, linear, leading to a behavior similar to the basic reciprocal toolkit function. There is a vertical intercept at (0, 6). This graph does not have a horizontal asymptote, since the degree of the numerator is larger than the degree of the denominator.

From the vertical intercept and horizontal intercept at x = -2, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.



From the graph, we can now tell on which intervals this expression will be non-negative, allowing the radical to be defined.
f(x) has domain $-2 \le x < 1$ *or* $x \ge 3$, or in interval notation, $[-2,1) \cup [3,\infty)$

Like with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as our concentration examples.

Example 6

The function $C(n) = \dfrac{20 + 0.4n}{100 + n}$ was used in the previous section to represent the concentration of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. We might want to be able to determine instead how much 40% solution has been added based on the current concentration of the mixture.

To do this, we would want the inverse of this function:

$C = \dfrac{20 + 0.4n}{100 + n}$ multiply up the denominator

$C(100 + n) = 20 + 0.4n$ distribute

$100C + Cn = 20 + 0.4n$ group everything with n on one side

$100C - 20 = 0.4n - Cn$ factor out n

$100C - 20 = (0.4 - C)n$ divide to find the inverse

$$n(C) = \frac{100C - 20}{0.4 - C}$$

If, for example, we wanted to know how many mL of 40% solution need to be added to obtain a concentration of 35%, we can simply evaluate the inverse rather than solving the original function:

$n(0.35) = \dfrac{100(0.35) - 20}{0.4 - 0.35} = \dfrac{15}{0.05} = 300$ mL of 40% solution would need to be added.

Try it Now

3. Find the inverse of the function $f(x) = \dfrac{x + 3}{x - 2}$

Important Topics of this Section

Imposing a coordinate system
Finding an inverse function
  Restricting the domain
Invertible toolkit functions
Rational Functions
Inverses of rational functions

Try it Now Answers

1. $x = f^{-1}(y) = \sqrt{y-1}$
2. identity, cubic, square root, cube root, exponential and logarithmic
3. $f^{-1}(y) = \dfrac{2y+3}{y-1}$

Section 3.5 Exercises

For each function, find a domain on which the function is one-to-one and non-decreasing, then find an inverse of the function on this domain.

1. $f(x) = (x-4)^2$
2. $f(x) = (x+2)^2$
3. $f(x) = 12 - x^2$
4. $f(x) = 9 - x^2$
5. $f(x) = 3x^3 + 1$
6. $f(x) = 4 - 2x^3$

Find the inverse of each function

7. $f(x) = 9 + \sqrt{4x-4}$
8. $f(x) = \sqrt{6x-8} + 5$
9. $f(x) = 9 + 2\sqrt[3]{x}$
10. $f(x) = 3 - \sqrt[3]{x}$
11. $f(x) = \dfrac{2}{x+8}$
12. $f(x) = \dfrac{3}{x-4}$
13. $f(x) = \dfrac{x+3}{x+7}$
14. $f(x) = \dfrac{x-2}{x+7}$
15. $f(x) = \dfrac{3x+4}{5-4x}$
16. $f(x) = \dfrac{5x+1}{2-5x}$

Police use the formula $v = \sqrt{20L}$ to estimate the speed of a car, v, in miles per hour, based on the length, L, in feet, of its skid marks when suddenly braking on a dry, asphalt road.

17. At the scene of an accident, a police officer measures a car's skid marks to be 215 feet long. Approximately how fast was the car traveling?

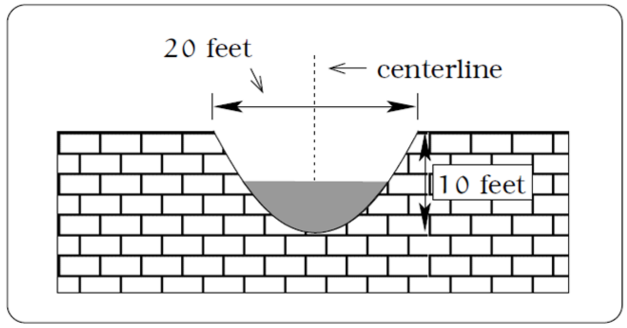
18. At the scene of an accident, a police officer measures a car's skid marks to be 135 feet long. Approximately how fast was the car traveling?

The formula $v = \sqrt{2.7r}$ models the maximum safe speed, v, in miles per hour, at which a car can travel on a curved road with radius of curvature r, in feet.

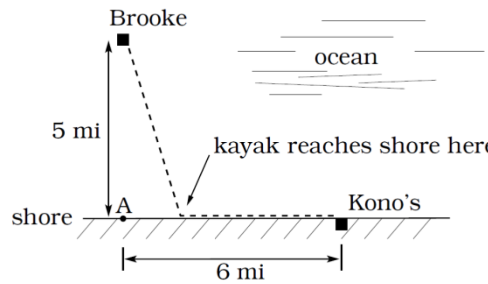
19. A highway crew measures the radius of curvature at an exit ramp on a highway as 430 feet. What is the maximum safe speed?

20. A highway crew measures the radius of curvature at a tight corner on a highway as 900 feet. What is the maximum safe speed?

21. A drainage canal has a cross-section in the shape of a parabola. Suppose that the canal is 10 feet deep and 20 feet wide at the top. If the water depth in the ditch is 5 feet, how wide is the surface of the water in the ditch? [UW]



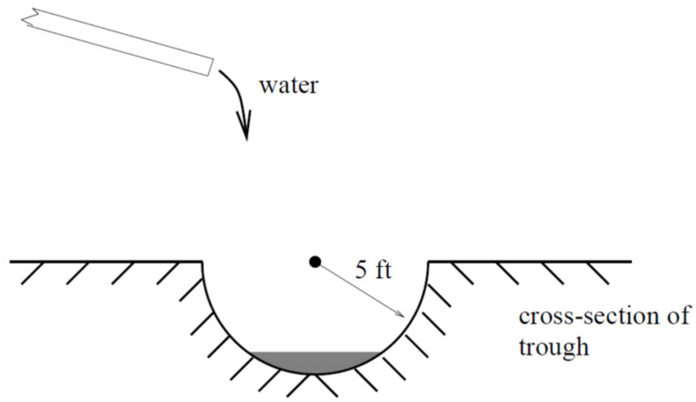
22. Brooke is located 5 miles out from the nearest point A along a straight shoreline in her seakayak. Hunger strikes and she wants to make it to Kono's for lunch; see picture. Brooke can paddle 2 mph and walk 4 mph. [UW]



   a. If she paddles along a straight line course to the shore, find an expression that computes the total time to reach lunch in terms of the location where Brooke beaches the boat.
   b. Determine the total time to reach Kono's if she paddles directly to the point A.
   c. Determine the total time to reach Kono's if she paddles directly to Kono's.
   d. Do you think your answer to b or c is the minimum time required for Brooke to reach lunch?
   e. Determine the total time to reach Kono's if she paddles directly to a point on the shore half way between point A and Kono's. How does this time compare to the times in parts b or c? Do you need to modify your answer to part d?

23. Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the x-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$. [UW]
   a. Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.
   b. Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?
   c. Clovis measures its height h of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.
   d. Does this function still work when the rocket is going down? Explain.

24. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour. [UW]



 a. Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.
 b. After how many hours will the surface of the water have width of 6 feet?
 c. Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

Chapter 4: Exponential and Logarithmic Functions

Section 4.1 Exponential Functions

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year[1]. We might ask if we can find a formula to model the population, P, as a function of time, t, in years after 2008, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by three each time the input increases by one. This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change. To see the significance of this difference consider these two companies:
Company A has 100 stores, and expands by opening 50 new stores a year
Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

Year	Stores, company A		Stores, company B
0	100	Starting with 100 each	100
1	100 + 50 = 150	They both grow by 50 stores in the first year.	100 + 50% of 100 100 + 0.50(100) = 150
2	150 + 50 = 200	Store A grows by 50, Store B grows by 75	150 + 50% of 150 150 + 0.50(150) = 225
3	200 + 50 = 250	Store A grows by 50, Store B grows by 112.5	225 + 50% of 225 225 + 0.50(225) = 337.5

[1] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 20, 2010

Notice that with the percent growth, each year, the company is growing by 50% of the current year total, so as the company grows larger, the number of stores added in a year grows as well.

To try to simplify the calculations, notice that after 1 year the number of stores for company *B* was:

$100 + 0.50(100)$ or equivalently by factoring

$100(1 + 0.50) = 150$

We can think of this as "the new number of stores is the original 100% plus another 50%"

After 2 years, the number of stores was:

$150 + 0.50(150)$ or equivalently by factoring

$150(1 + 0.50)$ now recall the 150 came from 100(1+0.50). Substituting that,

$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$

After 3 years, the number of stores was:

$225 + 0.50(225)$ or equivalently by factoring

$225(1 + 0.50)$ now recall the 225 came from $100(1 + 0.50)^2$. Substituting that,

$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.5$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of (1+0.50), so after *n* years, our equation would be

$B(n) = 100(1 + 0.50)^n$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential functions.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$f(x) = a(1 + r)^x$ or $f(x) = ab^x$ where $b = 1+r$

Where

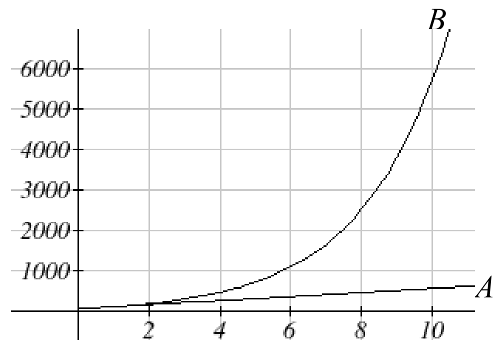
a is the initial or starting value of the function

r is the percent growth or decay rate, written as a decimal

b is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit *b* to positive values.

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company *A* and *B* described above over a longer time frame if the growth patterns were to continue

years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



Example 1

Write an exponential function for India's population, and use it to predict the population in 2020.

At the beginning of the chapter we were given India's population of 1.14 billion in the year 2008 and a percent growth rate of 1.34%. Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for $r = 0.0134$.

Using the basic formula for exponential growth $f(x) = a(1+r)^x$ we can write the formula, $f(t) = 1.14(1+0.0134)^t$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008.

$f(12) = 1.14(1+0.0134)^{12} \approx 1.337$ billion people in 2020

Try it Now

1. Given the three statements below, identify which one(s) is(are) exponential functions.

A. The cost of living allowance for state employees increases salaries by 3.1% each year.
B. State employees can expect a \$300 raise each year they work for the state.
C. Tuition costs have increased by 2.8% each year for the last 3 years.

Example 2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a \$1000 investment grow to over those 24 months?

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly. To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: 1.2%/12 = 0.1%. Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of $r = 0.001$, and our input m measured in months.

$$f(m) = 1000\left(1 + \frac{.012}{12}\right)^m$$

$$f(m) = 1000(1 + 0.001)^m$$

After 24 months, the account will have grown to $f(24) = 1000(1 + 0.001)^{24} = \$1024.28$

Try it Now

2. Looking at these two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

$A(t) = 1000(1.05)^t$ $\qquad$ $B(t) = 900(1.075)^t$

In all the preceding examples, we saw exponential growth. Exponential functions can also be used to model quantities that are decreasing at a percent rate. An example of this is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

Example 3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is a = 100 mg, and our growth rate will be negative 13%, since we are decreasing: r = -0.13. This gives the equation:

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be

$Q(7) = 100(0.87)^7 = 37.73$ mg of Bismuth-210 remains.

Try it Now

3. A population of 1000 is decaying 3% each year. Find the population in 30 years.

Example 4

T(q) represents the total number of Android smart phone contracts, in the thousands held by a certain Verizon store region measured quarterly since Jan 1st, 2010, Interpret all of the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan 1st, 2010 this region had 86,000 android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts are growing by 64%. $T(2) = 231.3056$ means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

Finding Equations of Exponential Functions

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function equation.

Example 5

In 2002, 80 deer were reintroduced into a wildlife refuge area from which the population had previously been hunted to elimination. By 2008, the population had grown to 180 deer. If this population grows exponentially, find a formula for the function.

By defining our input variable to be t, years after 2002, the information listed can be written as two input-output pairs: (0,80) and (6,180). Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: $a = 80$. This gives us an equation of the form

$$f(t) = 80b^t.$$

Substituting in our second input-output pair allows us to solve for b:

$$180 = 80b^6$$

$$b^6 = \frac{180}{80} = \frac{9}{4}$$

$$b = \sqrt[6]{\frac{9}{4}} = 1.1447$$

This gives us our equation for the population:

$$f(t) = 80(1.1447)^t$$

Recall that since $b = 1+r$, we can interpret this to mean that the population growth rate, $r = 0.1447$ and so the population is growing by about 14.47% each year.

In the previous example, we chose to use the $f(x) = ab^x$ form of the exponential function rather than the $f(x) = a(1+r)^x$ form. This choice was entirely arbitrary – either form would be fine to use.

When finding equations, the value for *b* or *r* will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 significant digits in the growth rate. For example, if your value for *b* was 1.00317643, you would want to round this no further than to 1.00318.

In the previous example, we were able to "give" ourselves the initial value by clever definition of our input variable. Next we consider the case where we can't do this.

Example 6

Find an equation for an exponential function passing through the points (-2,6) and (2,1)

Since we don't have the initial value, we will take a general approach that will work for any function form with unknown parameters: we will substitute in both given input-output pairs in the function form $f(x) = ab^x$ and solve for the unknown values, *a* & *b*.
Substituting in (-2, 6) gives $6 = ab^{-2}$
Substituting in (2, 1) gives $1 = ab^2$

We now solve these as a system of equations. To do so, we could try a substitution approach, solving one equation for a variable, then substituting that expression into the second equation.
Solving $6 = ab^{-2}$ for *a*:

$$a = \frac{6}{b^{-2}} = 6b^2$$

In the second equation, $1 = ab^2$, we substitute the expression above for *a*:

$$1 = (6b^2)b^2$$

$$1 = 6b^4$$

$$\frac{1}{6} = b^4$$

$$b = \sqrt[4]{\frac{1}{6}} \approx 0.6389$$

Going back to the equation $a = 6b^2$ lets us find *a*

$$a = 6b^2 = 6(0.6389)^2 = 2.4492$$

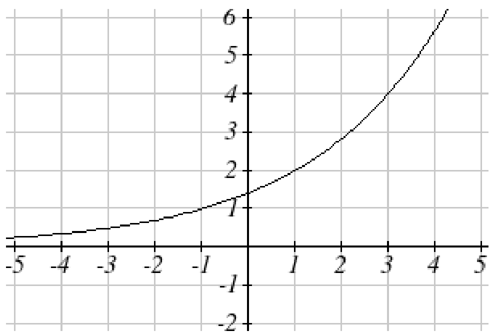
Putting this together gives the equation $f(x) = 2.4492(0.6389)^x$

Try it Now

4. Given the two points (1, 3) and (2, 4.5) find the equation of an exponential function that passes through these two points.

Example 7

Find an equation for the exponential function graphed below



The initial value for the function is not clear in this graph, so we will instead work using two clearer points. There are three fairly clear points: (-1, 1), (1, 2), and (3, 4). As we saw in the last example, two points are sufficient to find the equation for a standard exponential, so we will use the latter two points.

Substituting in (1,2) gives $2 = ab^1$

Substituting in (3,4) gives $4 = ab^3$

Solving the first equation for *a* gives $a = \frac{2}{b}$.

Substituting this expression for *a* into the second equation:

$$4 = ab^3$$

$$4 = \frac{2}{b}b^3 = \frac{2b^3}{b} \qquad \text{Simplify the right hand side}$$

$$4 = 2b^2$$

$$2 = b^2$$

$$b = \pm\sqrt{2}$$

Since we restrict ourselves to positive values of *b*, we will use $b = \sqrt{2}$. We can then go back and find *a*:

$$a = \frac{2}{b} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

This gives us a final equation of $f(x) = \sqrt{2}(\sqrt{2})^x$

Compound Interest
In the bank certificate of deposit (CD) example earlier in the section, we encountered compound interest. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, follow the pattern of compound interest. The term *compounding* comes from the behavior that interest is earned not only on the original value, but on the accumulated value of the account.

In the example from earlier, the interest was compounded monthly, so we took the annual interest rate, usually called the **nominal rate** or **annual percentage rate (APR)** and divided by 12, the number of compounds in a year, to find the monthly interest. The exponent was then measured in months.

Generalizing this, we can form a general equation for compound interest. If the APR is written in decimal form as r, and there are k compounds per year, then the interest per compounding period will be r/k. Likewise, if we are interested in the value after t years, then there will be kt compounding periods in that time.

Compound Interest Formula

Compound Interest can be calculated using the formula

$$A(t) = a\left(1+\frac{r}{k}\right)^{kt}$$

Where
$A(t)$ is the account value
t is measured in years
a is the starting amount of the account, often called the principal
r is the annual percentage rate (APR), also called the nominal rate
k is the number of compounds in one year

Example 8

If you invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Since we are starting with \$3000, $a = 3000$
Our interest rate is 3%, so $r = 0.03$
Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$
We want to know the value of the account in 10 years, we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1+\frac{0.03}{4}\right)^{4(10)} = \$4045.05$$

The account will be worth \$4045.05 in 10 years.

Example 9

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to $40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Since the account is earning 6%, $r = 0.06$
Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be $40,000, so we will be looking for the value of a so that $A(18) = 40,000$.

$$40{,}000 = A(18) = a\left(1+\frac{0.06}{2}\right)^{2(18)}$$

$$40{,}000 = a(2.8983)$$

$$a = \frac{40{,}000}{2.8983} \approx \$13{,}801$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Try it now

5. Recalculate example 2 from above with quarterly compounding

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If $1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

If we were to compute the actual percentage increase for the daily compounding, there was an increase of $105.16 from an original amount of $1,000, for a percentage increase of $\frac{105.16}{1000} = 0.10516 = 10.516\%$ increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year would be
$A(1) = a\left(1+\frac{r}{k}\right)^k$. To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount:

$$\frac{a\left(1+\frac{r}{k}\right)^k - a}{a} = \left(1+\frac{r}{k}\right)^k - 1$$

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APY = \left(1+\frac{r}{k}\right)^k - 1$$

Notice this is equivalent to finding the value of $1 after 1 year, and subtracting the original dollar.

Example 10

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield – the actual percent increase in a year.

Bank A: $APY = \left(1+\frac{0.012}{4}\right)^4 - 1 = 0.012054 = 1.2054\%$

Bank B: $APY = \left(1+\frac{0.011}{12}\right)^{12} - 1 = 0.011056 = 1.1056\%$

The monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

A Limit to Compounding

As we saw earlier, the amount we earn increases as we increase the compounding frequency. The table, though, shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	Value
Annual	$2
Semiannually	$2.25
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Minutely	$2.718279
Secondly	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number.

Euler's Number: *e*

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets big.

$e \approx 2.718282$

Since usually e is used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function *exp(x)*, where $exp(x) = e^x$.

Because e arises when compounding frequency gets big, e allows us to define **continuous growth** and is also one of our basic toolkit functions $f(x) = e^x$

Continuous Growth Equation

Continuous Growth can be calculated using the formula

$f(x) = ae^{rx}$

Where

a is the starting amount

r is the continuous growth rate

This type of equation is commonly used when describing quantities that change continuously, like chemical reactions, growth of large populations, and radioactive decay.

Example 11

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Since we are given a continuous decay rate, we use the continuous growth formula. Since we are decaying, we know the growth rate will be negative: r = -0.173
$f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

Try it Now

6. Interpret the following, $S(t) = 20e^{0.12t}$ if $S(t)$ represents the growth of a substance in grams, and time is measured in days.

Continuous growth is also often applied to compounded interest, allowing us to talk about continuous compounding.

The continuous growth rate is like the nominal growth rate – it reflects the growth rate before considering compounding. This is different than the annual growth rate used in the $f(x) = a(1+r)^x$, which is like the annual percentage yield – it reflects the *actual* amount the output grows in a year.

Example 12

If $1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Here, the continuous growth rate is 10%, so r = 0.10.
We start with $1000, so a = 1000.
To find the value after 1 year,
$f(1) = 1000e^{0.10(1)} \approx \$1105.17$

Notice that this value is slightly larger than the amount generated by daily compounding in the table computed earlier.

Important Topics of this Section

Percent growth
Exponential functions
- Finding equations
- Interpreting equations
- Graphs

Exponential Growth & Decay
Compounded interest
Annual Percent Yield
Continuous Growth

Try it Now Answers

1. A & C are exponential functions, they grow by a % not a constant number.
2. B(t) is growing faster, but after 3 years A(t) still has a higher account balance
3. $1000(0.97)^{30} = 401.0071$
4. $f(x) = 2(1.5)^x$
5. \$1024.25
6. An initial substance weighing 20g is growing at a continuous rate of 12% per day.

Section 4.1 Exercises

For each table below, could the table represent a function that is linear, exponential, or neither?

1.

x	1	2	3	4
f(x)	70	40	10	-20

2.

x	1	2	3	4
g(x)	40	32	26	22

3.

x	1	2	3	4
h(x)	70	49	34.3	24.01

4.

x	1	2	3	4
k(x)	90	80	70	60

5.

x	1	2	3	4
m(x)	80	61	42.9	25.61

6.

x	1	2	3	4
n(x)	90	81	72.9	65.61

7. A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population.

8. A population is currently 6,000 and has been increasing by 1.2% each day. Write an exponential model for the population.

9. The fox population in a certain region has an annual growth rate of 9 percent per year. It is estimated that the population in the year 2010 was 23,900. Estimate the fox population in the year 2018.

10. The amount of area covered by blackberry bushes in a park has been growing by 12% each year. It is estimated that the area covered in 2009 was 4,500 square feet. Estimate area that will be covered in 2020.

11. A vehicle purchased for $32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.

12. A business purchases $125,000 of office furniture which depreciates at a constant rate of 12% each year. Find the residual value of the furniture 6 years after purchase.

Find an equation for an exponential passing through the two points

13. $(0,6)$, $(3,750)$

14. $(0,3)$, $(2,75)$

15. $(0,2000)$, $(2,20)$

16. $(0,9000)$, $(3,72)$

17. $\left(-1,\frac{3}{2}\right),(3,24)$

18. $\left(-1,\frac{2}{5}\right),(1,10)$

19. $(-2,6),(3,1)$

20. $(-3,4)$, $(3,2)$

21. $(3,1)$, $(5,4)$

22. $(2,5)$, $(6,9)$

23. A radioactive substance decays exponentially. A scientist begins with 100 milligrams of a radioactive substance. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?

24. A radioactive substance decays exponentially. A scientist begins with 110 milligrams of a radioactive substance. After 31 hours, 55 mg of the substance remains. How many milligrams will remain after 42 hours?

25. A house was valued at $110,000 in the year 1985. The value appreciated to $145,000 by the year 2005. What was the annual growth rate between 1985 and 2005? Assume that the house value continues to grow by the same percentage. What will the value equal in the year 2010?

26. An investment was valued at $11,000 in the year 1995. The value appreciated to $14,000 by the year 2008. What was the annual growth rate between 1995 and 2008? Assume that the value continues to grow by the same percentage. What will the value equal in the year 2012?

27. A car was valued at $38,000 in the year 2003. The value depreciated to $11,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What will the value be in the year 2013?

28. A car was valued at $24,000 in the year 2006. The value depreciated to $20,000 by the year 2009. Assume that the car value continues to drop by the same percentage. What will the value be in the year 2014?

29. If 4000 dollars is invested in a bank account at an interest rate of 7 per cent per year, find the amount in the bank after 9 years if interest is compounded annually, quarterly, monthly, and continuously.

30. If 6000 dollars is invested in a bank account at an interest rate of 9 per cent per year, find the amount in the bank after 5 years if interest is compounded annually, quarterly, monthly, and continuously.

31. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 3% compounded quarterly.

32. Find the annual percentage yield (APY) for a savings account with annual percentage rate of 5% compounded monthly.

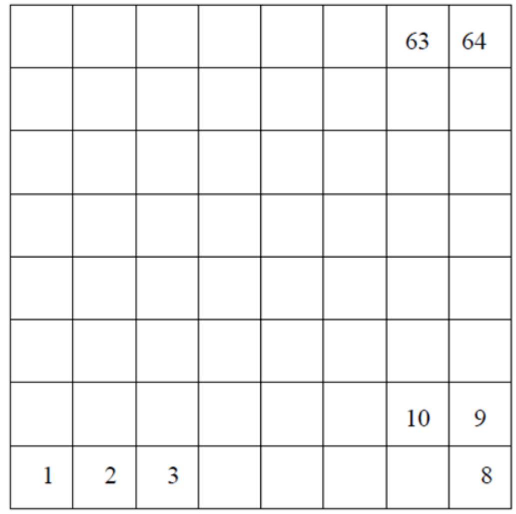
33. A population of bacteria is growing according to the equation $P(t)=1600e^{0.21t}$, with t measured in years. Estimate when the population will exceed 7569.

34. A population of bacteria is growing according to the equation $P(t)=1200e^{0.17t}$, with t measured in years. Estimate when the population will exceed 3443.

35. In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960. [UW]
    a. Find a formula for $w(t)$.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts.

36. In 1989, research scientists published a model for predicting the cumulative number of AIDS cases (in thousands) reported in the United States: $a(t)=155\left(\frac{t-1980}{10}\right)^3$, where t is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a new exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief." [UW]

37. You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always doubling the number from the previous square. [UW]



a. How many dimes will you have stacked on the 10th square?
b. How many dimes will you have stacked on the nth square?
c. How many dimes will you have stacked on the 64th square?
d. Assuming a dime is 1 mm thick, how high will this last pile be?
e. The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

Section 4.2 Graphs of Exponential Functions

Like with linear functions, the graph of an exponential function is determined by the values for the parameters in the equation in a logical way.

To get a sense for the behavior of exponentials, let us begin by looking more closely at the basic toolkit function $f(x) = 2^x$. Listing a table of values for this function:

x	-3	-2	-1	0	1	2	3
f(x)	1/8	¼	½	1	2	4	8

Notice that:

1) This function is positive for all values of *x*
2) As *x* increases, the function grows faster and faster
3) As *x* decreases, the function values grow smaller, approaching zero.
4) This is an example of exponential growth

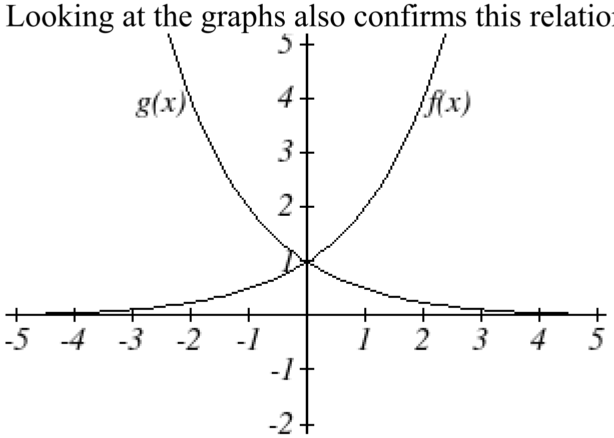
Looking at the function $g(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
g(x)	8	4	2	1	½	¼	1/8

Note this function is also positive for all values of *x*, but in this case grows as *x* decreases, and decreases towards zero as *x* increases. This is an example of exponential decay. You may notice from the table that this function appears to be the horizontal reflection of the $f(x) = 2^x$ table. This is in fact the case:

$$f(-x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x = g(x)$$

Looking at the graphs also confirms this relationship:



Since the initial value of the function is the function value at an input of zero, the initial value will give us the vertical intercept of the graph. From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the growth factor. This horizontal asymptote will also help us determine the long run behavior and is easy to see from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor b larger than one. When the growth rate is negative, the growth factor will be less than one.

Graphical Features of Exponential Functions

Graphically, in the function $f(x) = ab^x$
a is the vertical intercept of the graph
b determines the rate at which the graph grows
  the graph will increase if $b > 1$
  the graph will decrease if $0 < b < 1$
The graph will have a horizontal asymptote at $y = 0$

The domain of the function is all real numbers
The range of the function is $f(x) > 0$

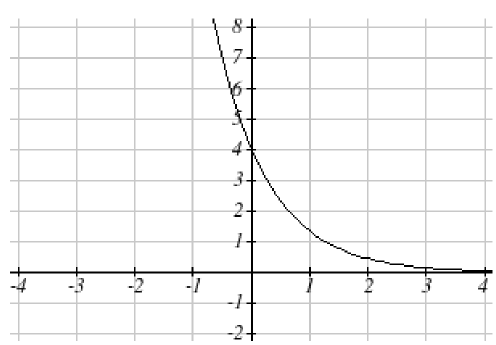
When sketching the graph of an exponential, it can be helpful to remember that the graph will pass through the points (0, a) and (1, ab)

The value b will determine the functions long run behavior.
If $b > 1$, as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to 0$.
If $0 < b < 1$, as $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to \infty$.

Example 1

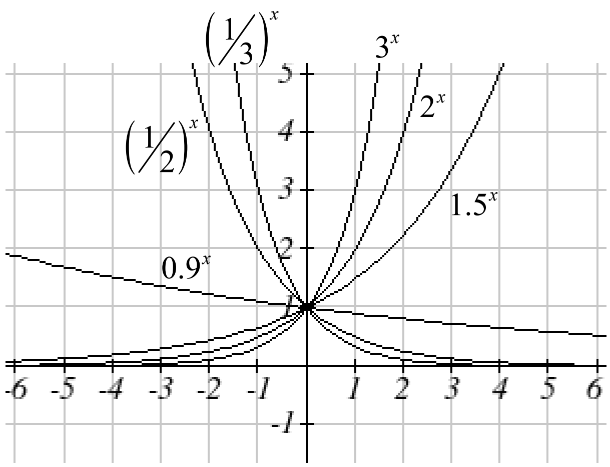
Sketch a graph of $f(x) = 4\left(\frac{1}{3}\right)^x$

This graph will have a vertical intercept at (0,4), and pass through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero.



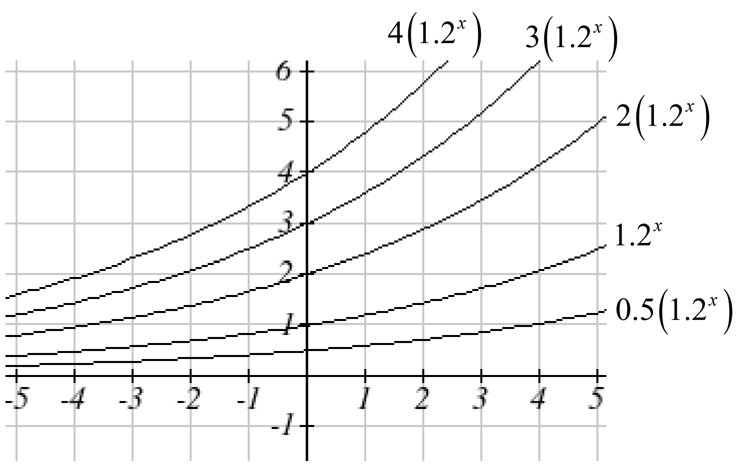
We can also see from the graph the long run behavior: as $x \to \infty$ the function $f(x) \to 0$ and as $x \to -\infty$ the function $f(x) \to \infty$.

To get a better feeling for the effect of *a* and *b* on the graph, examine the sets of graphs below. The first set shows various graphs, where *a* remains the same and we only change the value for *b*.



Notice that the closer the value of *b* is to 1, the flatter the graph will be.

In the next set of graphs, *a* is altered and our value for *b* remains the same.



Notice that changing the value for *a* changes the vertical intercept. Since *a* is multiplying the b^x term, *a* acts as a stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change.

Example 2

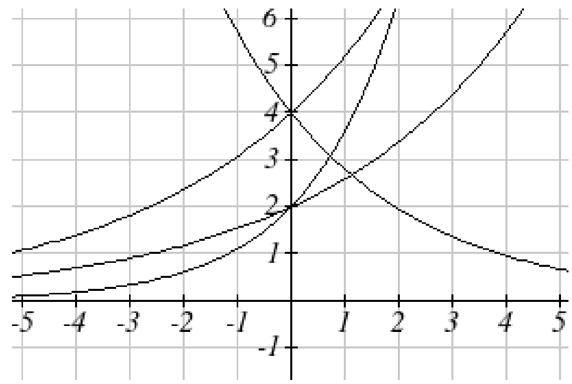
Match each equation with its graph.

$f(x) = 2(1.3)^x$

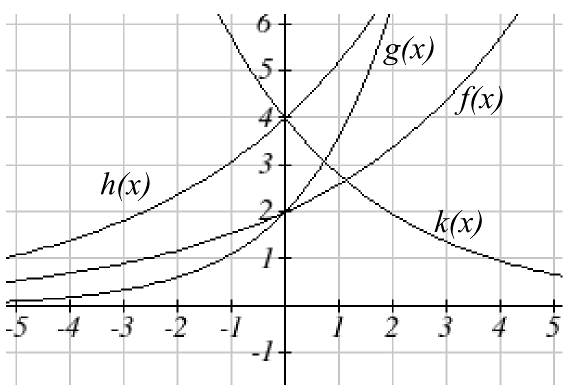
$g(x) = 2(1.8)^x$

$h(x) = 4(1.3)^x$

$k(x) = 4(0.7)^x$



The graph of *k(x)* is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of *h(x)* can be identified as the only growing exponential with a vertical intercept at (0,4). The graphs of *f(x)* and *g(x)* both have a vertical intercept at (0,2), but since *g(x)* has a larger growth factor, we can identify it as the graph increasing faster.



Try it Now

1. Graph the following functions on the same axis: $f(x) = (2)^x$; $g(x) = 2(2)^x$; $h(x) = 2(1/2)^x$.

Transformations of Exponential Graphs

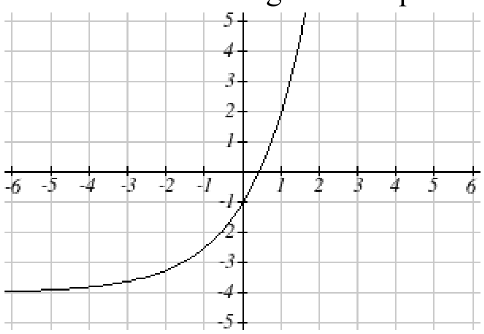
While exponential functions can be transformed following the same rules as any function, there are a few interesting features of transformations that can be identified. The first was seen at the beginning of the section – that a horizontal reflection is equivalent to a change in the growth factor. Likewise, since a is itself a stretch factor, a vertical stretch of an exponential is equivalent to a change in the initial value of the function.

Next consider the effect of a horizontal shift of an exponential. Shifting the function $f(x) = 3(2)^x$ four units to the left would give $f(x+4) = 3(2)^{x+4}$. Employing exponent rules, we could rewrite this:

$f(x+4) = 3(2)^{x+4} = 3(2)^x(2^4) = 48(2)^x$

Interestingly, it turns out that a horizontal shift of an exponential is equivalent to a change in initial value of the function.

Lastly, consider the effect of a vertical shift of an exponential. Shifting $f(x) = 3(2)^x$ down 4 units would give the equation $f(x) = 3(2)^x - 4$, yielding the graph



Notice that this graph is substantially different than the basic exponential graph. Unlike a basic exponential, this graph does not have a horizontal asymptote at $y = 0$; due to the vertical shift, the horizontal asymptote has also shifted to $y = -4$. We can see that as $x \to \infty$ the function $f(x) \to \infty$ and as $x \to -\infty$ the function $f(x) \to -4$.

From this, we have determined that a vertical shift is the only transformation of an exponential that changes the graph in a way unique from the effects of the basic parameters of an exponential

Transformations of Exponentials

Any transformed exponential can be written in the form

$f(x) = ab^x + c$

Where

c is the horizontal asymptote of the shifted exponential

Note that due to the shift, the vertical intercept is also shifted to $(0, a+c)$.

Try it Now

2. Write the equation and graph the exponential function described below;

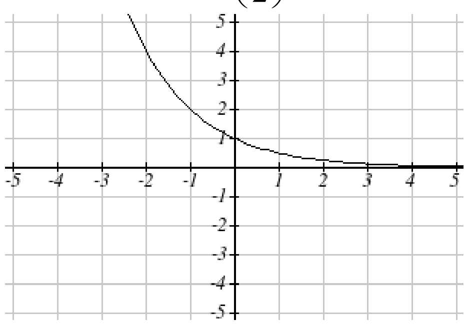
$f(x) = e^x$ is vertically stretched by a factor of 2, flipped across the y axis and shifted up 4 units.

Example 3

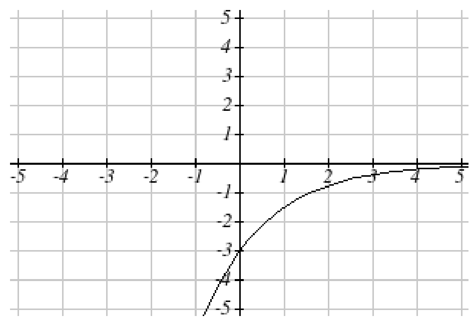
Sketch a graph of $f(x) = -3\left(\frac{1}{2}\right)^x + 4$

Notice that in this exponential, the negative in the stretch factor -3 will cause a vertical reflection of the graph, and the vertical shift up 4 will move the horizontal asymptote to $y = 4$. Sketching this as a transformation of a $g(x) = \left(\frac{1}{2}\right)^x$ graph,

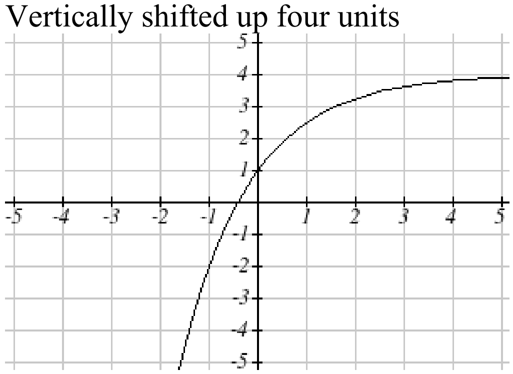
The basic $g(x) = \left(\frac{1}{2}\right)^x$



Vertically reflected and stretched by 3



Vertically shifted up four units

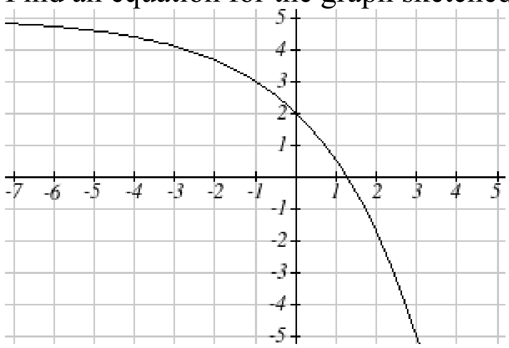


Notice that while the domain of this function is unchanged, due to the reflection and shift, the range of this function is $f(x) < 4$.
As $x \to \infty$ the function $f(x) \to 4$ and as $x \to -\infty$ the function $f(x) \to -\infty$

Equations leading to graphs like the one above are common as models for learning models and models of growth approaching a limit.

Example 4

Find an equation for the graph sketched below



Looking at this graph, it appears to have a horizontal asymptote at $y = 5$, suggesting an equation of the form $f(x) = ab^x + 5$. To find values for *a* and *b*, we can identify two other points on the graph. It appears the graph passes through (0,2) and (-1,3), so we can use those points. Substituting in (0,2) allows us to solve for *a*

$$2 = ab^0 + 5$$

$$2 = a + 5$$

$$a = -3$$

Substituting in (-1,3) allows us to solve for *b*

$$3 = -3b^{-1} + 5$$

$$-2 = \frac{-3}{b}$$

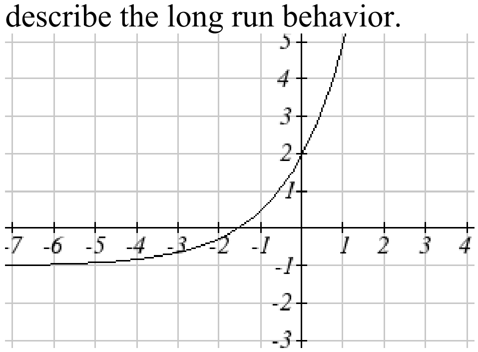
$$-2b = -3$$

$$b = \frac{3}{2} = 1.5$$

The final equation for our graph is $f(x) = -3(1.5)^x + 5$

Try it Now

3. Given the graph of the transformed exponential function, write the equation and describe the long run behavior.

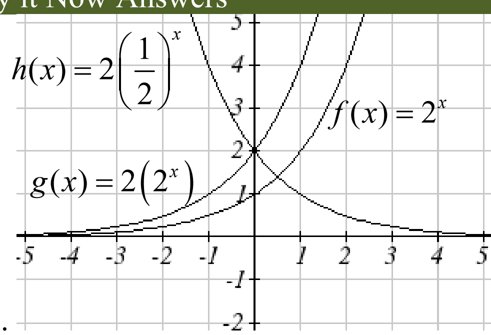


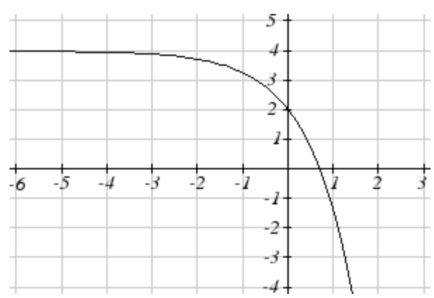
Important Topics of this Section

Graphs of exponential functions
- Intercept
- Growth factor

Exponential Growth
Exponential Decay
Horizontal intercepts
Long run behavior
Transformations

Try it Now Answers

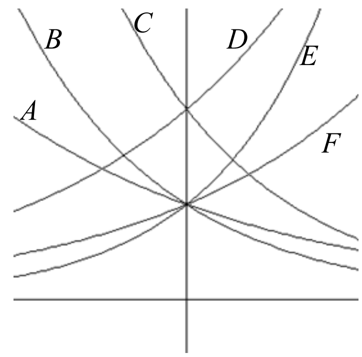
1. 

2. $f(x)=-2e^x+4$; 

3. $f(x)=3(.5)^{-x}-1$ or $f(x)=3(2^x)-1$; As $x\to\infty$ the function $f(x)\to\infty$ and as $x\to-\infty$ the function $f(x)\to-1$

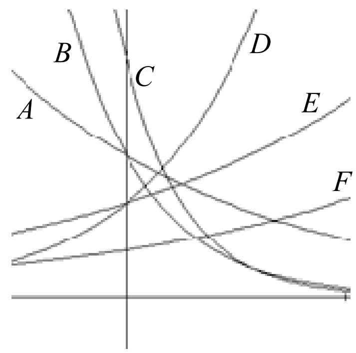
Section 4.2 Exercises

Match each equation with one of the graphs below



1. $f(x) = 2(0.69)^x$
2. $f(x) = 2(1.28)^x$
3. $f(x) = 2(0.81)^x$
4. $f(x) = 4(1.28)^x$
5. $f(x) = 2(1.59)^x$
6. $f(x) = 4(0.69)^x$

If all the graphs to the right have equations with form $f(x) = ab^x$



7. Which graph has the largest value for b?
8. Which graph has the smallest value for b?
9. Which graph has the largest value for a?
10. Which graph has the smallest value for a?

Sketch a graph of each of the following transformations of $f(x) = 2^x$

11. $f(x) = 2^{-x}$
12. $g(x) = -2^x$
13. $h(x) = 2^x + 3$
14. $f(x) = 2^x - 4$
15. $f(x) = 2^{x-2}$
16. $k(x) = 2^{x-3}$

Starting with the graph of $f(x) = 4^x$, write the equation of the graph that results from

17. Shifting $f(x)$ 4 units upwards
18. Shifting $f(x)$ 3 units downwards
19. Shifting $f(x)$ 2 units left
20. Shifting $f(x)$ 5 units right
21. Reflecting $f(x)$ about the x-axis
22. Reflecting $f(x)$ about the y-axis

Describe the long run behavior, as $x \to \infty$ and $x \to -\infty$ of each function

23. $f(x) = -5(4^x) - 1$

24. $f(x) = -2(3^x) + 2$

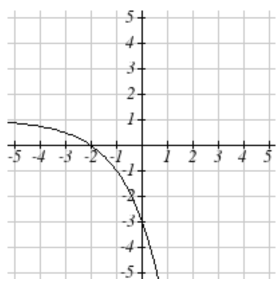
25. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

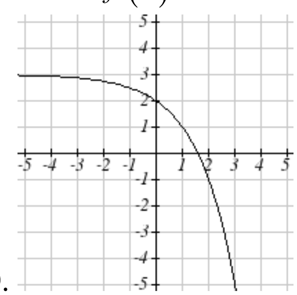
26. $f(x) = 4\left(\frac{1}{4}\right)^x + 1$

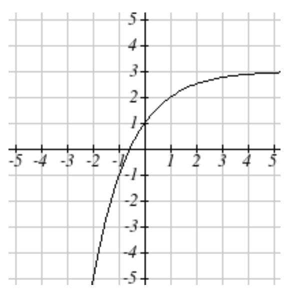
27. $f(x) = 3(4)^{-x} + 2$

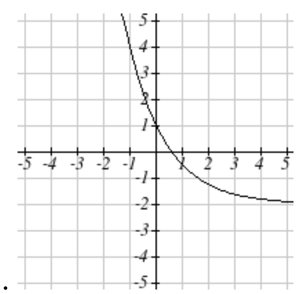
28. $f(x) = -2(3)^{-x} - 1$

Find an equation for each graph as a transformation of $f(x) = 2^x$

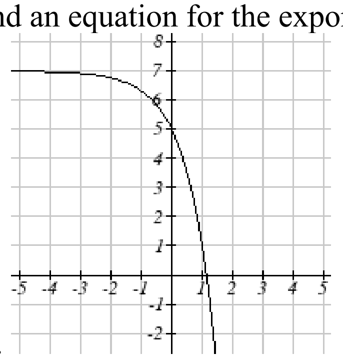
29. 

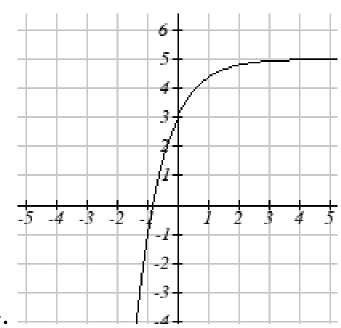
30. 

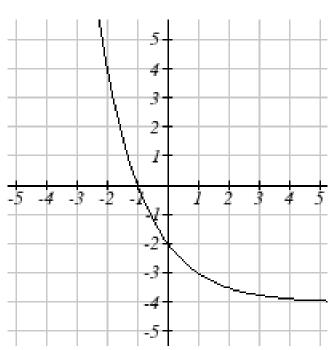
31. 

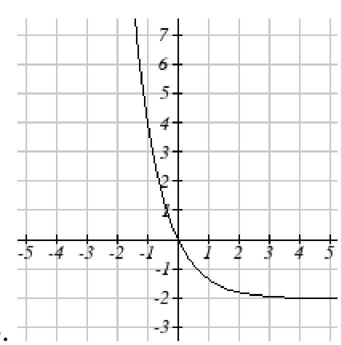
32. 

Find an equation for the exponential graphed.

33. 

34. 

35. 

36. 

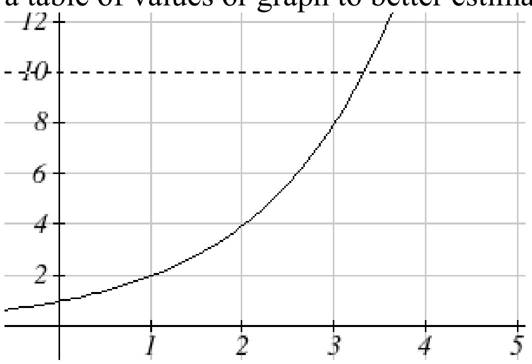
Section 4.3 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to an equation of the form $f(x) = 50(2)^x$. When will this population reach 500? Trying to solve this problem leads to

$500 = 50(2)^x$

$10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4. We could use technology to create a table of values or graph to better estimate the solution.



From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, and since the exponential function is one-to-one, it would be great to have an inverse function. None of the functions we have already discussed would serve as an inverse function and so we must introduce a new function, named **log** as the inverse of an exponential function. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.

Logarithm

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b).

Since the logarithm and exponential are inverses, it follows that:

Properties of Logs: Inverse Properties

$\log_b(b^x) = x$

$b^{\log_b x} = x$

Recall also that from the definition of an inverse function that if $f(a) = c$, then $f^{-1}(c) = a$. Applying this to the exponential and logarithmic functions:

Logarithm Equivalent to an Exponential

The statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$

Alternatively, we could show this by starting with the exponential function $c = b^a$, then taking the log base *b* of both sides, giving $\log_b(c) = \log_b b^a$. Using the inverse property of logs we see that $\log_b(c) = a$.

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluate, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 1

Write these exponential expressions as logarithmic expressions:

$2^3 = 8$ $\quad$ $5^2 = 25$ $\quad$ $10^{-4} = \frac{1}{10000}$

$2^3 = 8$ is equivalent to $\log_2(8) = 3$

$5^2 = 25$ is equivalent to $\log_5(25) = 2$

$10^{-4} = \frac{1}{10000}$ is equivalent to $\log_{10}\left(\frac{1}{10000}\right) = -4$

Example 2

Write these logarithmic expressions as exponential expressions

$\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ $\quad$ $\log_3(9) = 2$

$\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to $6^{1/2} = \sqrt{6}$

$\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try it Now

Write the exponential expression $4^2 = 16$ as a logarithm.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 3

Solve $\log_4(x) = 2$ for *x*.

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$

Example 4

Solve $2^x = 10$ for *x*.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we had made to the solution earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$

The **natural log** is the logarithm with base *e*, and is typically written $\ln(x)$

Example 5

Evaluate $\log(1000)$ using the definition of the common log.

To evaluate $\log(1000)$, we can say $x = \log(1000)$, then rewrite into exponential form using the common log base of 10.

$10^x = 1000$

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to write $\log_{10} 10^3 = 3$

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Try it Now

2. Evaluate $\log(1000000)$

Example 6

Evaluate $\ln(\sqrt{e})$ using the definition of the natural log.

To evaluate $\ln(\sqrt{e})$, we can say $x = \ln(\sqrt{e})$. Rewriting as an exponential, $e^x = \sqrt{e}$. You may recall that the square root is equivalent to a power of ½ so x = ½.

Example 7

Evaluate log(500) using your calculator or computer.

Using a computer, we can evaluate $\log(500) \approx 2.69897$

To utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need to establish some additional properties.

Properties of Logs: Exponent Property

$\log_b\left(A^r\right) = r\log_b(A)$

To show why this is true, we offer a proof.
Since the logarithm and exponential are inverses, $b^{\log_b A} = A$.
So $A^r = \left(b^{\log_b A}\right)^r$
Utilizing the exponential rule that states $\left(x^a\right)^b = x^{ab}$,
$A^r = \left(b^{\log_b A}\right)^r = b^{r\log_b A}$

So then $\log_b\left(A^r\right) = \log_b\left(b^{r\log_b A}\right)$
Again utilizing the inverse property on the right side yields the result
$\log_b\left(A^r\right) = r\log_b A$

Example 8

Rewrite $\log_3(25)$ using the exponent property for logs.

Since $25 = 5^2$,
$\log_3(25) = \log_3\left(5^2\right) = 2\log_3 5$

Example 9

Rewrite $4\ln(x)$ using the exponent property for logs

Using the property in reverse, $4\ln(x)=\ln\left(x^4\right)$

Try it Now

3. Rewrite using the exponent property for logs: $\ln\left(\frac{1}{x^2}\right)$

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$$\log_b(A)=\frac{\log_c(A)}{\log_c(b)}$$

Proof.

Let $\log_b(A)=x$. Rewriting as an exponential gives $b^x=A$. Taking the log base *c* of both sides of this equation gives

$\log_c b^x=\log_c A$

Now utilizing the exponent property for logs on the left side,

$x\log_c b=\log_c A$

Dividing, we obtain

$x=\frac{\log_c A}{\log_c b}$ or replacing our expression for *x*, $\log_b A=\frac{\log_c A}{\log_c b}$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

Example 10

Evaluate $\log_2(10)$ using the change of base formula.

According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base *e*

$$\log_2 10=\frac{\log_e 10}{\log_e 2}=\frac{\ln 10}{\ln 2}$$

Using our calculators to evaluate this,

$$\frac{\ln 10}{\ln 2} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

Example 11

Evaluate $\log_5(100)$ using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we were able to solve the basic exponential equation $2^x = 10$ by rewriting in exponential form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 12

Solve $2^x = 10$ for x.

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$\ln\left(2^x\right) = \ln(10)$ Utilizing the exponent property for logs,

$x\ln\left(2\right) = \ln(10)$ Now dividing by ln(2),

$$x = \frac{\ln(10)}{\ln\left(2\right)}$$

Notice that this result is equivalent to the result we found using the change of base formula.

Example 13

In the first section, we predicted the population (in billions) of India *t* years after 2008 by the equation $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for the *t* so that *f(t)* = 2

$2 = 1.14(1.0134)^t$ Divide by 1.14 to isolate the exponential expression

$\frac{2}{1.14} = 1.0134^t$ Take the logarithm of both sides of the equation

$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$ Apply the exponent property on the right side

$\ln\left(\frac{2}{1.14}\right) = t\ln\left(1.0134\right)$ Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln\left(1.0134\right)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Try it Now

4. Solve $5(0.93)^x = 10$

In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 14

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, H^+, measured in Moles, by the equation $pH = -\log\left(H^+\right)$.

If a liquid has concentration of 0.0001 Moles, determine the pH.

Determine the hydrogen concentration of a liquid with pH of 7.

To answer the first question, we evaluate the expression $-\log\left(0.0001\right)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$-\log\left(0.0001\right) = -\log\left(10^{-4}\right) = -(-4) = 4$

To answer the second question, we need to solve the equation $7 = -\log\left(H^+\right)$. Begin by isolating the logarithm on one side of the equation by dividing by a negative.

$-7 = \log\left(H^+\right)$

Now rewriting into exponential form yields the answer

$H^+ = 10^{-7} = 0.0000001$ Moles

Logarithms also provide us a mechanism for finding continuous growth equations for exponentials given two points.

Example 15

A population of beetles grows from 100 to 130 in 2 weeks. Find the continuous growth rate.

Measuring *t* is weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0) = 100$ and $P(2) = 130$. Using the first pair of values,

$100 = ae^{r0}$, so $a = 100$.

Using the second pair of values,

$130 = 100e^{r2}$ Divide by 100

$\frac{130}{100} = e^{r2}$ Take the natural log of both sides

$\ln(1.3) = \ln\left(e^{r2}\right)$ Use the inverse property of logs

$\ln(1.3) = 2r$

$r = \frac{\ln(1.3)}{2} \approx 0.1312$

This population is growing at a continuous rate of 13.12% per week.

In general, we can relate the standard form of an exponential with the continuous growth form by noting (using *k* to represent the continuous growth rate to avoid the confusion of using *r* twice in two different ways in the same formula)

$a(1+r)^x = ae^{kx}$

$(1+r)^x = e^{kx}$

$1 + r = e^k$

Using this, we see that it is always possible to convert from the continuous growth form of an exponential to the standard form and vice versa.

Example 16

A company's sales have been growing following the function $S(t) = 5000e^{0.12t}$. Find the annual growth rate.

Noting that $1 + r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form $S(t) = 5000(1 + 0.1275)^t$

Important Topics of this Section

The Logarithmic function as the inverse of the exponential function
Writing logarithmic & exponential expressions
Properties of logs
- Inverse properties
- Exponential properties
- Change of base

Common log
Natural log
Solving exponential equations

Try it Now Answers

1. $\log_4(16) = 2 = \log_4 4^2 = 2\log_4 4$
2. 6
3. $-2\ln(x)$
4. $\dfrac{\ln(2)}{\ln(0.93)} \approx -9.5513$

Section 4.3 Exercises

Rewrite each equation in exponential form

1. $\log_4(q) = m$
2. $\log_3(t) = k$
3. $\log_a(b) = c$
4. $\log_p(z) = u$
5. $\log(v) = t$
6. $\log(r) = s$
7. $\ln(w) = n$
8. $\ln(x) = y$

Rewrite each equation in logarithmic form.

9. $4^x = y$
10. $5^y = x$
11. $c^d = k$
12. $n^z = L$
13. $10^a = b$
14. $10^p = v$
15. $e^k = h$
16. $e^y = x$

Solve for *x*.

17. $\log_3(x) = 2$
18. $\log_4(x) = 3$
19. $\log_2(x) = -3$
20. $\log_5(x) = -1$
21. $\log(x) = 3$
22. $\log(x) = 5$
23. $\ln(x) = 2$
24. $\ln(x) = -2$

Simplify each expression using logarithm properties

25. $\log_5(25)$
26. $\log_2(8)$
27. $\log_3\left(\frac{1}{27}\right)$
28. $\log_6\left(\frac{1}{36}\right)$
29. $\log_6\left(\sqrt{6}\right)$
30. $\log_5\left(\sqrt[3]{5}\right)$
31. $\log(10{,}000)$
32. $\log(100)$
33. $\log(0.001)$
34. $\log(0.00001)$
35. $\ln\left(e^{-2}\right)$
36. $\ln\left(e^3\right)$

Evaluate using your calculator

37. $\log(0.04)$
38. $\log(1045)$
39. $\ln(15)$
40. $\ln(0.02)$

Solve each equation for the variable

41. $5^x = 14$
42. $3^x = 23$
43. $7^x = \frac{1}{15}$
44. $3^x = \frac{1}{4}$
45. $e^{5x} = 17$
46. $e^{3x} = 12$
47. $3^{4x-5} = 38$
48. $4^{2x-3} = 44$
49. $1000(1.03)^t = 5000$
50. $200(1.06)^t = 550$
51. $3(1.04)^{3t} = 8$
52. $2(1.08)^{4t} = 7$
53. $50e^{-0.12t} = 10$
54. $10e^{-0.03t} = 4$
55. $10 - 8\left(\frac{1}{2}\right)^x = 5$
56. $100 - 100\left(\frac{1}{4}\right)^x = 70$

Convert the equation into continuous growth $f(t) = ae^{kt}$ form

57. $f(t) = 300(0.91)^t$

58. $f(t) = 120(0.07)^t$

59. $f(t) = 10(1.04)^t$

60. $f(t) = 1400(1.12)^t$

Convert the equation into annual growth $f(t) = ab^t$ form

61. $f(t) = 150e^{0.06t}$

62. $f(t) = 100e^{0.12t}$

63. $f(t) = 50e^{-0.012t}$

64. $f(t) = 80e^{-0.85t}$

65. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

66. The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million?

67. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. If the population continues to grow exponentially at the same rate, when will the population exceed 1 million people?

68. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. If it continues to grow exponentially at the same rate, when will median income exceed $50,000?

69. A scientist begins with 100 mg of a radioactive substance. After 4 hours, it has decayed to 80 mg. How long will it take to decay to 15 mg?

70. A scientist begins with 100 mg of a radioactive substance. After 6 days, it has decayed to 60 mg. How long will it take to decay to 10 mg?

71. If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500?

72. If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300?

Section 4.4 Logarithmic Properties

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

Properties of Logs

Inverse Properties:

$$\log_b\left(b^x\right)=x$$

$$b^{\log_b x}=x$$

Exponential Property:

$$\log_b\left(A^r\right)=r\log_b\left(A\right)$$

Change of Base:

$$\log_b\left(A\right)=\frac{\log_c(A)}{\log_c(b)}$$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems in exponential and logarithmic equations.

Properties of Logs

Sum of Logs Property:

$$\log_b\left(A\right)+\log_b\left(C\right)=\log_b\left(AC\right)$$

Difference of Logs Property:

$$\log_b\left(A\right)-\log_b\left(C\right)=\log_b\left(\frac{A}{C}\right)$$

As an important note, the logarithm represents a function and does *not* follow regular algebraic distribution rules that you may be used to. The "word log" does not distribute into parenthesis, and so you must learn these new rules.

To help in this process we offer a proof to help solidify our new rules and show how they follow from properties you've already seen.

Let $a=\log_b\left(A\right)$ and $c=\log_b\left(C\right)$, so by definition of the logarithm, $b^a=A$ and $b^c=C$

Using these expressions, $AC = b^a b^c$
Using exponent rules on the right, $AC = b^{a+c}$
Taking the log of both sides, and utilizing the inverse property of logs,
$\log_b(AC) = \log_b\left(b^{a+c}\right) = a + c$
Replacing *a* and *c* with their definition establishes the result
$\log_b(AC) = \log_b A + \log_b C$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or a break an expression involving a single log into expressions involving multiple logs.

Example 1

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Using the sum of logs property on the first two terms,
$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$

This reduces our original expression to $\log_3(40) - \log_3(2)$

Then using the difference of logs property,
$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Example 2

Evaluate $2\log(5) + \log(4)$ without a calculator by first rewriting as a single logarithm.

On the first term, we can use the exponent property of logs to write
$2\log(5) = \log\left(5^2\right) = \log(25)$

With the expression reduced to a sum of two logs, $\log(25) + \log(4)$, we can utilize the sum of logs property
$\log(25) + \log(4) = \log(4 \cdot 25) = \log(100)$

Since $100 = 10^2$, we can evaluate this log without a calculator:
$\log(100) = \log\left(10^2\right) = 2$

Try it Now

1. Without a calculator evaluate by first rewriting as a single logarithm
$\log_2(8) + \log_2(4)$

Example 3

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs

First noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4 y}{7}\right) = \ln\left(x^4 y\right) - \ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln\left(x^4 y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Interestingly, solving exponential equations was not the reason logarithms were originally developed. Historically, up until the advent of calculators and computers, the power of logarithms was that these log properties allowed multiplication, division, roots, and powers to be evaluated using addition and subtraction, which is much easier to compute without a calculator. Large books of logarithm values were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the sum of log properties were used. Suppose for example we didn't know the value of 2 times 3. Using the sum property of logs

value	log(value)
1	0.0000000
2	0.3010300
3	0.4771213
4	0.6020600
5	0.6989700
6	0.7781513
7	0.8450980
8	0.9030900
9	0.9542425
10	1.0000000

$$\log(2 \cdot 3) = \log(2) + \log(3)$$

Using the log table,

$$\log(2 \cdot 3) = \log(2) + \log(3) = 0.3010300 + 0.4771213 = 0.7781513$$

We can then use the table again in reverse, looking for 0.7781513 as the result of the log. From that we can determine

$$\log(2 \cdot 3) = 0.7781513 = \log(6)$$

By doing addition and the table of logs, we were able to determine $2 \cdot 3 = 6$.

Likewise, to compute a cube root like $\sqrt[3]{8}$

$$\log(\sqrt[3]{8}) == \log\left(8^{1/3}\right) = \frac{1}{3}\log(8) = \frac{1}{3}(0.9030900) = 0.3010300 = \log(2)$$

So $\sqrt[3]{8} = 2$

Although these calculations are simple and insignificant they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

These properties still have practical applications for interpreting changes in exponential and logarithmic relationships.

Example 4

Recall that in chemistry, $pH = -\log\left(H^+\right)$. If the concentration of hydrogen ions in a liquid is doubled, what is the affect on pH?

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P = -\log\left(C\right)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log\left(2C\right)$$

Using the sum property of logs,

$$pH = -\log\left(2C\right) = -\left(\log(2) + \log(C)\right) = -\log(2) - \log(C)$$

Since $P = -\log\left(C\right)$, the new pH is

$$pH = P - \log(2) = P - 0.301$$

After the concentration of hydrogen ions is doubled, the pH will decrease by 0.301.

Log properties in solving equations
The logarithm properties often arise when solving problems involving logarithms

Example 5

Solve $\log(50x + 25) - \log(x) = 2$

In order to rewrite as an exponential, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x + 25}{x}\right) = 2$$

Rewriting in exponential form reduces this to an algebraic equation

$$\frac{50x + 25}{x} = 10^2 = 100$$

Solving,

$50x + 25 = 100x$

$25 = 50x$

$x = \frac{25}{50} = \frac{1}{2}$

Try it Now

2. Solve $\log(x^2 - 4) = 1 + \log(x + 2)$

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 6a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year[2]. If these trends continue, when will the population of Kenya match that of Sudan?

We start by writing an equation for each population in terms of *t*, years after 2008.

$Kenya(t) = 38.8(1 + 0.264)^t$

$Sudan(t) = 41.3(1 + 0.224)^t$

To find when the populations will be equal, we can set the equations equal

$38.8(1.264)^t = 41.3(1.224)^t$

For our first approach, we take the log of both sides of the equation

$\log\left(38.8(1.264)^t\right) = \log\left(41.3(1.224)^t\right)$

Utilizing the sum property of logs, we can rewrite each side,

$\log(38.8) + \log\left(1.264^t\right) = \log(41.3) + \log\left(1.224^t\right)$

Then utilizing the exponent property, we can pull the variables out of the exponent

$\log(38.8) + t\log(1.264) = \log(41.3) + t\log(1.224)$

Moving all the terms involving *t* to one side of the equation and the rest of the terms to the other side,

$t\log(1.264) - t\log(1.224) = \log(41.3) - \log(38.8)$

[2] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010

Factoring out the t on the left,
$t(\log(1.264) - \log(1.224)) = \log(41.3) - \log(38.8)$

Dividing to solve for t
$t = \dfrac{\log(41.3) - \log(38.8)}{\log(1.264) - \log(1.224)} \approx 1.942$ years until the populations will be equal

Example 6b

Solve the problem above using rewriting before taking the log

Starting at the equation
$38.8(1.264)^t = 41.3(1.224)^t$

Divide to move the exponential terms to one side of the equation and the constants to the other side
$\dfrac{1.264^t}{1.224^t} = \dfrac{41.3}{38.8}$

Using exponent rules to group on the left,
$\left(\dfrac{1.264}{1.224}\right)^t = \dfrac{41.3}{38.8}$

Taking the log of both sides
$\log\left(\left(\dfrac{1.264}{1.224}\right)^t\right) = \log\left(\dfrac{41.3}{38.8}\right)$

Utilizing the exponent property on the left,
$t\log\left(\dfrac{1.264}{1.224}\right) = \log\left(\dfrac{41.3}{38.8}\right)$

Dividing gives
$t = \dfrac{\log\left(\dfrac{41.3}{38.8}\right)}{\log\left(\dfrac{1.264}{1.224}\right)} \approx 1.942$ years

While the answer does not immediately appear identical to that produced using the previous method, note that by using the different property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.264}{1.224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.264) - \log(1.224)}$$

While both methods work equally well, it often requires less steps to utilize algebra before taking logs, rather than relying on log properties.

Try it Now

3. Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Important Topics of this Section

Inverse
Exponential
Change of base
Sum of logs property
Difference of logs property
Solving equations using log rules

Try it Now Answers

1. 5
2. 12
3. 4.1874 weeks

Section 4.4 Exercises

Simplify using logarithm properties to a single logarithm

1. $\log_3(28) - \log_3(7)$
2. $\log_3(32) - \log_3(4)$
3. $-\log_3\left(\frac{1}{7}\right)$
4. $-\log_4\left(\frac{1}{5}\right)$
5. $\log_3\left(\frac{1}{10}\right) + \log_3(50)$
6. $\log_4(3) + \log_4(7)$
7. $\frac{1}{3}\log_7(8)$
8. $\frac{1}{2}\log_5(36)$
9. $\log\left(2x^4\right) + \log\left(3x^5\right)$
10. $\ln\left(4x^2\right) + \ln\left(3x^3\right)$
11. $\ln\left(6x^9\right) - \ln\left(3x^2\right)$
12. $\log\left(12x^4\right) - \log(4x)$
13. $2\log(x) + 3\log(x+1)$
14. $3\log(x) + 2\log\left(x^2\right)$
15. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$
16. $2\log(x) + \frac{1}{3}\log(y) - \log(z)$

Use logarithm properties to expand each expression

17. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
18. $\log\left(\frac{a^2b^3}{c^5}\right)$
19. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
20. $\ln\left(\frac{a^{-2}b^3}{c^{-5}}\right)$
21. $\log\left(\sqrt{x^3y^{-4}}\right)$
22. $\log\left(\sqrt{x^{-3}y^2}\right)$
23. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
24. $\ln\left(\frac{x}{\sqrt{1-x^2}}\right)$
25. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$
26. $\log\left(x^3y^4\sqrt[7]{x^3y^9}\right)$

Solve each equation for the variable

27. $4^{4x-7} = 3^{9x-6}$
28. $2^{2x-5} = 7^{3x-7}$
29. $17(1.14)^x = 19(1.16)^x$
30. $20(1.07)^x = 8(1.13)^x$
31. $5e^{0.12t} = 10e^{0.08t}$
32. $3e^{0.09t} = e^{0.14t}$
33. $\log_2(7x+6) = 3$
34. $\log_3(2x+4) = 2$
35. $2\ln(3\text{x}) + 3 = 1$
36. $4\ln(5x) + 5 = 2$
37. $\log(x^3) = 2$
38. $\log(x^5) = 3$
39. $\log(x) + \log(x+3) = 3$
40. $\log(x+4) + \log(x) = 9$
41. $\log(x+4) - \log(x+3) = 1$
42. $\log(x+5) - \log(x+2) = 2$
43. $\log_6(x^2) - \log_6(x+1) = 1$
44. $\log_3(x^2) - \log_3(x+2) = 5$
45. $\log(x+12) = \log(x) + \log(12)$
46. $\log(x+15) = \log(x) + \log(15)$
47. $\ln(x) + \ln(x-3) = \ln(7x)$
48. $\ln(x) + \ln(x-6) = \ln(6x)$

Section 4.5 Graphs of Logarithmic Functions

Recall that the exponential function $f(x) = 2^x$ would produce this table of values

x	-3	-2	-1	0	1	2	3
f(x)	1/8	¼	½	1	2	4	8

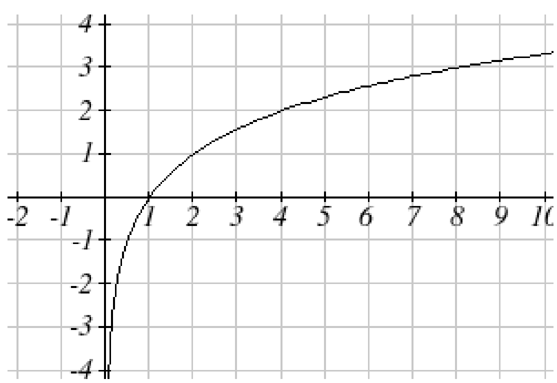
Since the logarithmic function is an inverse of the exponential, $g(x) = \log_2 x$ would produce the table of values

x	1/8	¼	½	1	2	4	8
g(x)	-3	-2	-1	0	1	2	3

Notice that

1) As the input increases, the output increases
2) As *x* increases, the output decreases more slowly
3) Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs.
4) Since the exponential function can accept all real numbers as inputs, the logarithm can output any real number
5) We can also recall from our study of toolkit functions that the domain if the logarithmic function is $(0, \infty)$ and the range is all real numbers or $(-\infty, \infty)$

Sketching the graph,



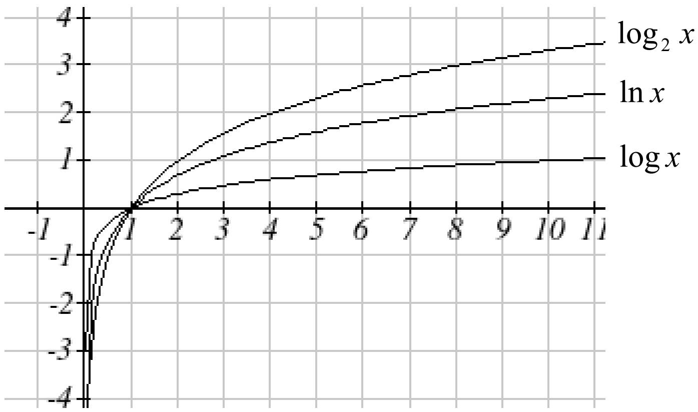
Notice that as the input approaches zero from the right, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.
In symbolic notation we write as
$x \to 0^+, f(x) \to -\infty$, and
as $x \to \infty, f(x) \to \infty$

Graphical Features of the Logarithm

Graphically, in the function $g(x) = \log_b x$
The graph has a horizontal intercept at (1, 0)
The graph has a vertical asymptote at $x = 0$
The domain of the function is $x > 0$ or $(0, \infty)$
The range of the function is all real numbers $(-\infty, \infty)$

When sketching a general logarithm, it can be helpful to remember that the graph will pass through the points (1, 0) and (*b*, 1)

To get a feeling for how the base affects the shape of the graph, examine the sets of graphs below.



Notice that the larger the base, the slower the graph will grow. For example, the common log graph, while it can grow as large as you'd like, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Another important observation made was the domain of the logarithm. Along with division and the square root, the logarithm is a function that restricts the domain of a function.

Example 1

Find the domain of the function $f(x) = \log(5 - 2x)$

The logarithm is only defined with the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality,

$-2x > -5$

$x < \dfrac{5}{2}$

The domain of this function is $x < \dfrac{5}{2}$, or in interval notation, $\left(-\infty, \dfrac{5}{2}\right)$

Try it Now

1. Find the domain of the function $f(x) = \log(x - 5) + 2$, before solving this as an inequality, consider how the function has been transformed.

Transformations of the Logarithmic Function
Like with exponentials, transformations can be done using the basic transformation techniques, but several transformations have interesting relations.

First recall the change of base property tells us that $\log_b x = \dfrac{\log_c x}{\log_c b} = \dfrac{1}{\log_c b}\log_c x$

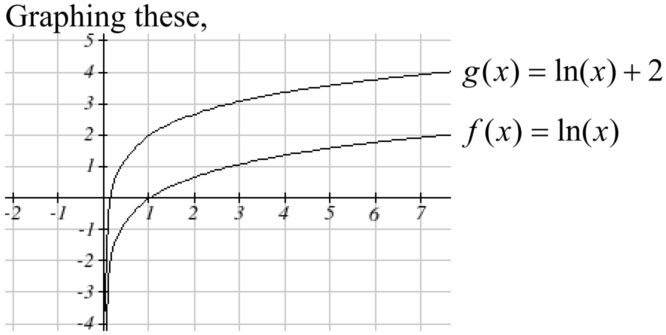
From this, we can see that $\log_b x$ is a vertical stretch or compression of the graph of the $\log_c x$ graph. This tells us that a vertical stretch or compression is equivalent to a change of base. For this reason, we typically represent all graphs of logarithmic functions in terms of the common or natural log functions.

Next, consider the effect of a horizontal compression on the graph of a logarithmic function. Considering $f(x) = \log(cx)$, we can use the sum property to see
$f(x) = \log(cx) = \log(c) + \log(x)$

Since log(c) is a constant, the effect of a horizontal compression is the same as the effect of a vertical shift. To see what this effect looks like,

Example 2
Sketch $f(x) = \ln(x)$ and $g(x) = \ln(x) + 2$

Graphing these,



Note that as we saw, this vertical shift could also be written as a horizontal compression:
$g(x) = \ln(x) + 2 = \ln(x) + \ln(e^2) = \ln(e^2 x)$

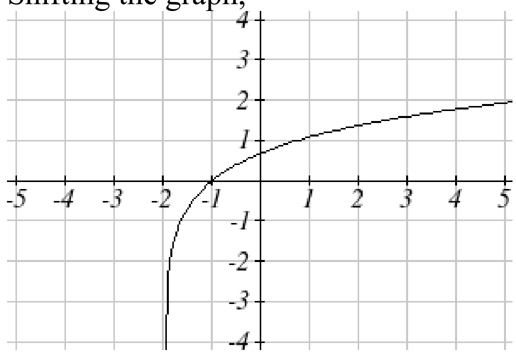
While a horizontal stretch or compression can be written as a vertical shift, a horizontal reflection is unique and separate from vertical shifting.

Finally, we will consider the effect of a horizontal shift on the graph of a logarithm

Example 3

Sketch a graph of $f(x)=\ln(x+2)$

This is a horizontal shift to the left by 2 units. Notice that none of our logarithm rules allow us rewrite this in another form, so the effect of this transformation is unique. Shifting the graph,



Notice that due to the horizontal shift, the vertical asymptote shifted as well, to x = -2

Combining these transformations,

Example 4

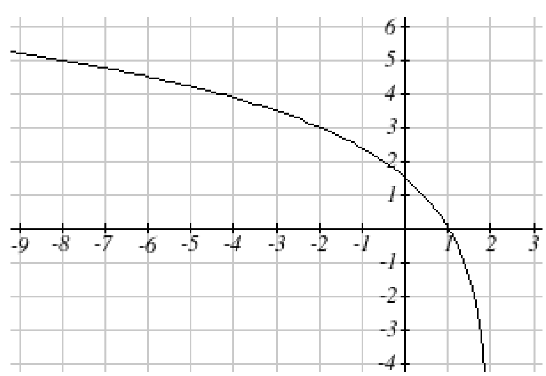
Sketch a graph of $f(x)=5\log(-x+2)$

Factoring the inside as $f(x)=5\log(-(x-2))$ reveals that this graph is that of the common logarithm, horizontally reflected, vertically stretched by a factor of 5, and shifted to the right by 2 units.

The vertical asymptote will have been shifted to $x = 2$, and the graph will be defined for $x < 2$. A rough sketch can be created by using the vertical asymptote along with a couple points on the graph, such as

$f(1)=5\log(-1+2)=5\log(1)=0$

$f(-8)=5\log(-(-8)+2)=5\log(10)=5$

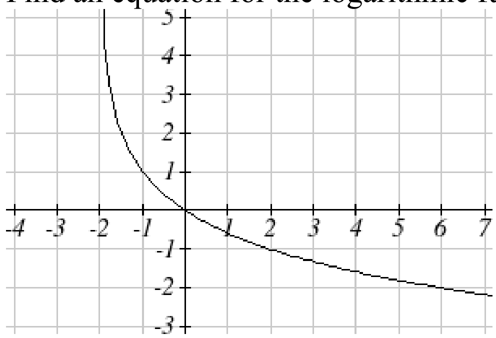


Try it Now

2. Sketch a graph of the function $f(x)=-3\log(x-2)+1$

Example 5

Find an equation for the logarithmic function graphed below



This graph has a vertical asymptote at x = -2 and has been vertically reflected. We do not know yet the vertical shift (equivalent to horizontal stretch) or the vertical stretch (equivalent to a change of base). We know so far that the equation will have form $f(x) = -a\log(x+2)+k$

It appears the graph passes through the points (-1,1) and (2,-1). Substituting in (-1,1),

$1 = -a\log(-1+2)+k$

$1 = -a\log(1)+k$

$1 = k$

Next substituting in (2,-1),

$-1 = -a\log(2+2)+1$

$-2 = -a\log(4)$

$a = \dfrac{2}{\log(4)}$

This gives us the equation $f(x) = -\dfrac{2}{\log(4)}\log(x+2)+1$

Flashback

3. Using the graph above write the Domain & Range and describe the long run behavior.

Important Topics of this Section

Graph of the logarithmic function (domain & range)
Transformation of logarithmic functions
Creating graphs from equations
Creating equations from graphs

Try it Now Answers

1. Domain: {x| x > 5}
2. Input a graph of $f(x) = -3\log(x-2)+1$

Flashback Answers

3. Domain: {x|x>-2}, Range: All Real Numbers; As $x \to -2^+, f(x) \to \infty$ and as $x \to \infty, f(x) \to -\infty$

Section 4.5 Exercises

For each function, find the domain and the vertical asymptote

1. $f(x) = \log(x-5)$
2. $f(x) = \log(x+2)$
3. $f(x) = \ln(3-x)$
4. $f(x) = \ln(5-x)$
5. $f(x) = \log(3x+1)$
6. $f(x) = \log(2x+5)$
7. $f(x) = 3\log(-x)+2$
8. $f(x) = 2\log(-x)+1$

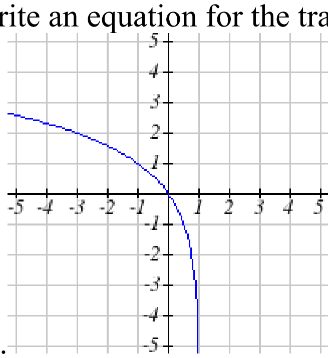
Sketch a graph of each pair of function

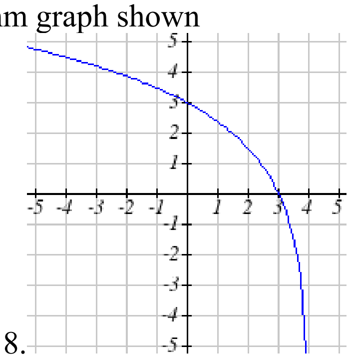
9. $f(x) = \log(x), g(x) = \ln(x)$
10. $f(x) = \log_2(x), g(x) = \log_4(x)$

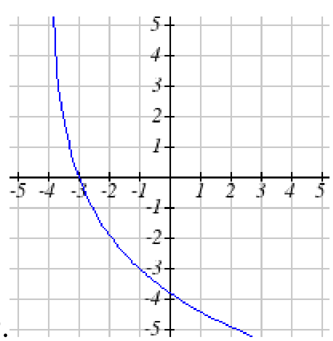
Sketch each transformation

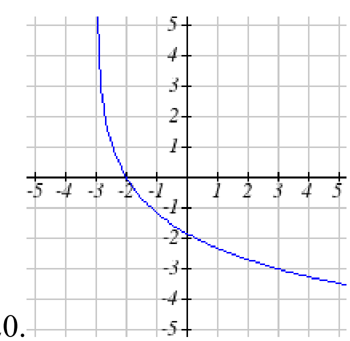
11. $f(x) = 2\log(x)$
12. $f(x) = 3\ln(x)$
13. $f(x) = \ln(-x)$
14. $f(x) = -\log(x)$
15. $f(x) = \log_2(x+2)$
16. $f(x) = \log_3(x+4)$

Write an equation for the transformed logarithm graph shown

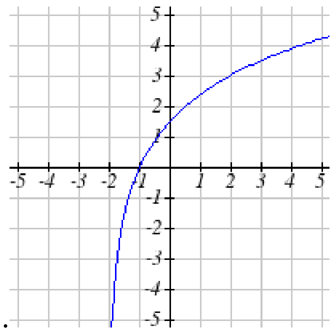
17. 

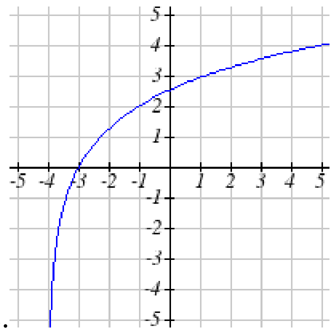
18. 

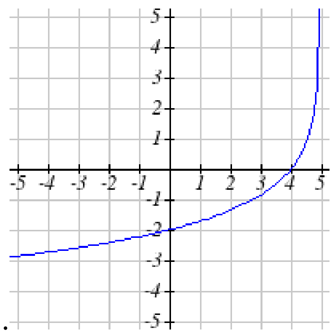
19. 

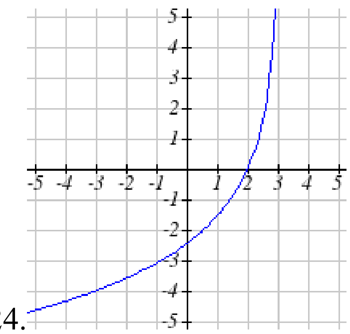
20. 

Write an equation for the transformed logarithm graph shown

21. 

22. 

23. 

24. 

Section 4.6 Exponential and Logarithmic Models

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

Radioactive Decay
In an earlier section, we discussed radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

Half Life

The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation $h(t) = ab^t$, half life can be found by solving for when half the original amount remains – by solving $\frac{1}{2}a = a(b)^t$, or more simply $\frac{1}{2} = b^t$. Notice how the initial amount is irrelevant when solving for half life

Example 1

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant *a*. Then the decay of Bismuth-210 can be described by the equation $Q(d) = a(0.87)^d$.

To find the half-life, we want to determine when the remaining quantity is half the original: ½*a*. Solving,

$\frac{1}{2}a = a(0.87)^d$ Dividing by *a*,

$\frac{1}{2} = 0.87^d$ Take the log of both sides

$\log\left(\frac{1}{2}\right) = \log\left(0.87^d\right)$ Use the exponent property of logs

$\log\left(\frac{1}{2}\right) = d\log(0.87)$ Divide to solve for *d*

$$d = \frac{\log\left(\frac{1}{2}\right)}{\log(0.87)} = 4.977 \text{ days}$$

This tells us that the half-life of Bismuth-210 is approximately 5 days.

Example 2

Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Since the half-life is 30 years, after 30 years, half the original amount, 100mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25mg.

Example 3

Cesium-137 has a half-life of about 30 years. Find the annual decay rate.

Since we are looking for an annual growth rate, we will use an equation of the form $Q(t) = a(1+r)^t$. We know that after 30 years, half the original amount will remain. Using this information

$\frac{1}{2}a = a(1+r)^{30}$ Dividing by *a*

$\frac{1}{2} = (1+r)^{30}$ Taking the 30th root of both sides

$\sqrt[30]{\frac{1}{2}} = 1+r$ Subtracting one from both sides,

$$r = \sqrt[30]{\frac{1}{2}} - 1 \approx -0.02284$$

This tells us cesium-137 is decaying at an annual rate of 2.284% per year.

Try it Now

Chlorine-36 is eliminated from the body with a biological half-life of 10 days[3]. Find the daily decay rate.

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[3] http://www.ead.anl.gov/pub/doc/chlorine.pdf

Example 4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula – we will use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate,

$\frac{1}{2}a = ae^{r5730}$ Dividing by a

$\frac{1}{2} = e^{r5730}$ Taking the natural log of both sides

$\ln\left(\frac{1}{2}\right) = \ln\left(e^{r5730}\right)$ Use the inverse property of logs on the right side

$\ln\left(\frac{1}{2}\right) = 5730r$ Divide by 5730

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Now we know the decay will follow the equation $Q(t) = ae^{-0.000121t}$. To find how old the bone fragment is that contains 20% of the original amount, we solve for t so that $Q(t) = 0.20a$.

$$0.20a = ae^{-0.000121t}$$
$$0.20 = e^{-0.000121t}$$
$$\ln(0.20) = \ln\left(e^{-0.000121t}\right)$$
$$\ln(0.20) = -0.000121t$$
$$t = \frac{\ln(0.20)}{-0.000121} \approx 13301 \text{ years}$$

The bone fragment is about 13301 years old.

Try it Now

2. In example 2 we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Doubling Time
For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

Doubling Time

The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation $h(t) = ab^t$, doubling time can be found by solving for when the original quantity has doubled - by solving $2a = a(b)^x$, or more simply $2 = b^x$. Again notice how the initial amount is irrelevant when solving for doubling time.

Example 5

Cancerous cells can grow exponentially. If a cancerous growth contained 300 cells last month and 360 cells this month, how long will it take for the number of cancerous cells to double?

Defining t to be time in months, with $t = 0$ corresponding to this month, we are given two pieces of data: this month, (0, 360), and last month, (-1, 300).

From this data, we can find an equation for the growth. Using the form $C(t) = ab^t$, we know immediately $a = 360$, giving $C(t) = 360b^t$. Substituting in (-1, 300),

$$300 = 360b^{-1}$$
$$300 = \frac{360}{b}$$
$$b = \frac{360}{300} = 1.2$$

This gives us the equation $C(t) = 360(1.2)^t$

To find the doubling time, we look for the time until we have twice the original amount, so when $C(t) = 2a$.

$$2a = a(1.2)^t$$
$$2 = (1.2)^t$$
$$\log(2) = \log\left(1.2^t\right)$$
$$\log(2) = t\log(1.2)$$
$$t = \frac{\log(2)}{\log(1.2)} \approx 3.802 \text{ years.}$$

It will take about 3.8 years for the number of cancerous cells to double.

Example 6

A new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If they currently have 120 thousand users and this trend continues, how many users will the site have in 1 year?

We can use the doubling time to find an equation for the growth of the site, and then use that equation to answer the question. While we could use an arbitrary *a* as we have before for the initial amount, in this case, we know the initial amount was 120 thousand.

If we use a continuous growth equation, it would look like $N(t) = 120e^{rt}$, measured in thousands of users after *t* months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$
$$2 = e^{r5}$$
$$\ln 2 = 5r$$
$$r = \frac{\ln 2}{5} \approx 0.1386$$

Now that we have an equation, $N(t) = 120e^{0.1386t}$, we can predict the number of users after 12 months:

$N(12) = 120e^{0.1386(12)} = 633.140$ thousand users.

So after 1 year, we would expect the site to have around 633,140 users.

Try it Now

3. If tuition is increasing by 6.6% each year, how many years will it take to tuition to double?

Newton's Law of Cooling

When a hot object is left in surrounding air that is lower temperature, the object's temperature will decrease exponentially, leveling off towards the surrounding air temperature. Since the graph levels off at the surrounding air temperature, the equation must have a horizontal asymptote at this value, meaning the equation for a decaying exponential must have been shifted up.

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$T(t) = ae^{kt} + T_s$

Where

t is time

a is a constant determined by the initial temperature of the object

k is a constant, the continuous rate of cooling of the object

While an equation of the form $T(t) = ab^t + T_s$ could be used, the continuous form is more common.

Example 7

A cheesecake is taken out of the oven with an ideal internal temperature of 165 degrees Fahrenheit, and is placed into a 35 degree refrigerator. After 10 minutes, the cheesecake has cooled to 150 degrees. If you must wait until the cheesecake has cooled to 70 degrees before you eat it, how long will you have to wait?

Since the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially towards 35, following the equation

$T(t) = ae^{kt} + 35$

We know the initial temperature was 165, so $T(0) = 165$. Substituting in these values,

$165 = ae^{k0} + 35$

$165 = a + 35$

$a = 130$

We were given another pair of data, $T(10) = 150$, which we can use to solve for k

$150 = 130e^{k10} + 35$

$115 = 130e^{k10}$

$\frac{115}{130} = e^{10k}$

$\ln\left(\frac{115}{130}\right) = 10k$

$k = \frac{\ln\left(\frac{115}{130}\right)}{10} = -0.0123$

Together this gives us the equation for cooling: $T(t) = 130e^{-0.0123t} + 35$

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{-0.0123t} + 35$$

$$35 = 130e^{-0.0123t}$$

$$\frac{35}{130} = e^{-0.0123t}$$

$$\ln\left(\frac{35}{130}\right) = -0.0123t$$

$$t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68$$

It will take about 107 minutes, or a little over an hour and half, for the cheesecake to cool enough to be eaten.

Try it Now

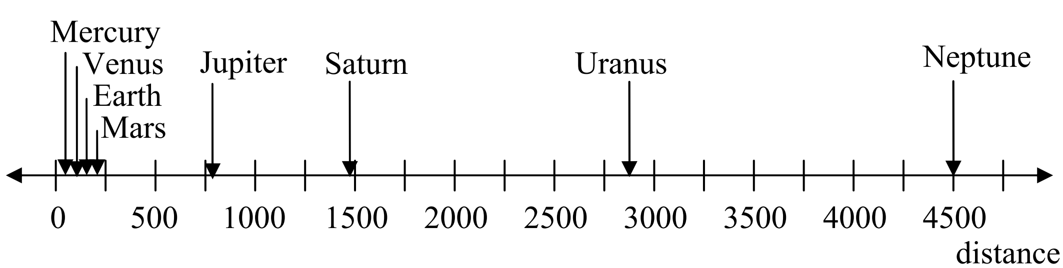
4. A thermos of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

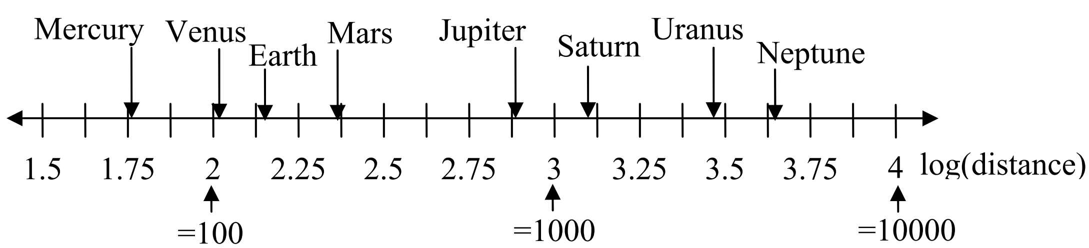
Planet	**Distance (millions of km)**
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up.

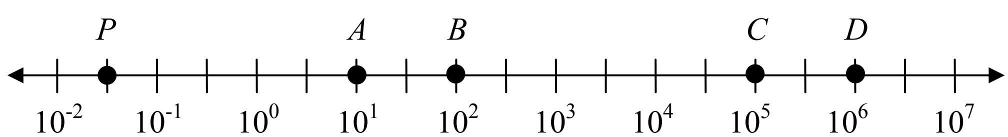


However, by taking the logarithm of these values makes the values more manageable. Placing these values on a number line by their log values makes the relative distances more apparent.

Planet	**Distance (millions of km)**	**log(distance)**
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65



Sometimes a log scale will show the logarithm of values, but more commonly the values are listed, sometimes as powers of 10 as in the scale here



Example 8

Estimate the value of point P on the log scale above

The point P appears to be half way between -2 and -1 in log value, so if V is the value of this point,

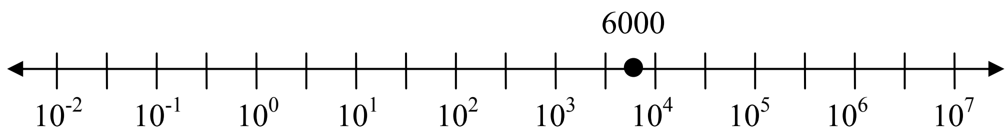
$\log(V) \approx -1.5$ Rewriting in exponential form,

$V \approx 10^{-1.5} = 0.0316$

Example 9

Place the number 6000 on a logarithmic scale.

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:



Try it Now

5. Plot the data in the table below on a logarithmic scale[4]

Source of Sound/Noise	Approximate Sound Pressure in μPa (micro Pascals)
Launching of the Space Shuttle	2,000,000,000
Full Symphony Orchestra	2,000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

Notice that on a log scale from above, the visual distance on the scale between points *A* and *B* and between *C* and *D* is the same. When looking at the values these points correspond to, notice *B* is ten times the value of *A*, and *D* is ten times the value of *C*. A visual *linear* difference between points corresponds to a *relative* (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing $1,000,000 to $10,000, the first is 100 times larger than the second.

$$\frac{1{,}000{,}000}{10{,}000} = 100 = 10^2$$

Likewise, comparing $1000 to $10, the first is 100 times larger than the second.

$$\frac{1{,}000}{10} = 100 = 10^2$$

When one quantity is ten times larger than another, we say it is one **order of magnitude** larger. In both these cases, the first number was two orders of magnitude larger than the second.

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[4] From http://www.epd.gov.hk/epd/noise_education/web/ENG_EPD_HTML/m1/intro_5.html, retrieved Oct 2, 2010

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, *B* is one order of magnitude larger than *A*, and *D* is one order of magnitude larger than *C*.

Orders of Magnitude

Given two values *A* and *B*, to determine how many **orders of magnitude** *B* is greater than *A*,

Difference in orders of magnitude = $\log\left(\frac{A}{B}\right)$

Example 10

On the log scale above, how many orders of magnitude larger is *C* than *B*.

The value *B* corresponds to $10^2 = 100$
The value *C* corresponds to $10^5 = 100{,}000$

The relative change is $\frac{100{,}000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3. *C* is three orders of magnitude greater than *B*, which can be seen on the log scale by the visual difference between the points on the scale.

Try it Now

6. Using the table from Try it Now #5, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle.

An example of a logarithmic scale is the Moment Magnitude Scale (MMS) used for earthquakes. This scale is commonly and mistakenly called the Richter Scale, which was a very similar scale succeeded by the MMS.

Moment Magnitude Scale

For an earthquake with seismic moment *S*, a measurement of earth movement, the MMS value, or magnitude of the earthquake, is

$$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

Where S_0 is a baseline measure for the seismic moment. $S_0 = 10^{16}$

Example 11

If one earthquake has a MMS magnitude of 6.0, and another has a magnitude of 8.0, how much more powerful – more earth movement – does the second earthquake have?

Since the first earthquake has magnitude 6.0, we can find the amount of earth movement. The value of S_0 is not particularly relevant, so we will not replace it with its value.

$$6.0 = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

$$6.0\left(\frac{3}{2}\right) = \log\left(\frac{S}{S_0}\right)$$

$$9 = \log\left(\frac{S}{S_0}\right)$$

$$\frac{S}{S_0} = 10^9$$

$$S = 10^9 S_0$$

Doing the same with the second earthquake with a magnitude of 8.0,

$$8.0 = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

$$S = 10^{12} S_0$$

From this, we can see that this second value's earth movement is 1000 times as large as the first earthquake.

Example 12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Since the first quake has magnitude 3.0,

$$3.0 = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

$$3.0\left(\frac{3}{2}\right) = \log\left(\frac{S}{S_0}\right)$$

$$4.5 = \log\left(\frac{S}{S_0}\right)$$

$$10^{4.5} = \frac{S}{S_0}$$

$$S = 10^{4.5} S_0$$

Since the second earthquake has twice as much earth movement, for the second quake, $S = 2 \cdot 10^{4.5} S_0$

Finding the magnitude,

$$M = \frac{2}{3}\log\left(\frac{2 \cdot 10^{4.5} S_0}{S_0}\right)$$

$$M = \frac{2}{3}\log\left(2 \cdot 10^{4.5}\right) \approx 3.201$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.2.

In fact, using log properties, we could show that whenever the earth movement doubles, the magnitude will increase by about 0.201:

$$M = \frac{2}{3}\log\left(\frac{2S}{S_0}\right) = \frac{2}{3}\log\left(2 \cdot \frac{S}{S_0}\right)$$

$$M = \frac{2}{3}\left(\log(2) + \log\left(\frac{S}{S_0}\right)\right)$$

$$M = \frac{2}{3}\log(2) + \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

$$M = 0.201 + \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$

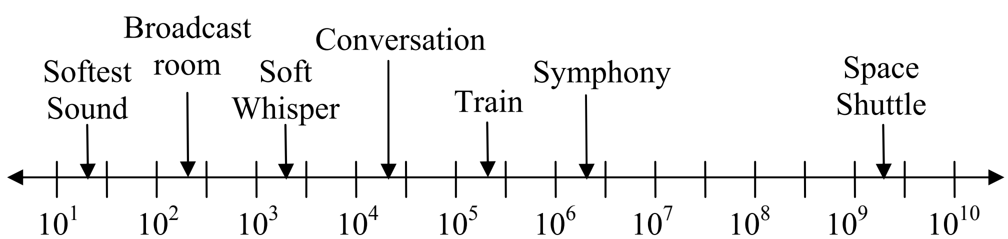
This illustrates the most important feature of a log scale: that *multiplying* the quantity being considered will *add* to the scale value, and vice versa.

Important Topics of this Section

Radioactive decay
Half life
Doubling time
Newton's law of cooling
Logarithmic Scales
Orders of Magnitude
Moment Magnitude scale

Try it Now Answers

1. $r = \sqrt[10]{\frac{1}{2}} - 1 \approx -0.067$ or 6.7% is the daily rate of decay.
2. Less than 230 years, 229.3157 to be exact
3. 10.845 years or approximately 11 years tuition will have doubled
4. 6.026 hours
5.



6. $\frac{2x10^9}{2x10^1} = 10^8$ The sound pressure in µPa created by launching the space shuttle is 8 orders of magnitude greater than the sound pressure in µPa created by the softest sound a human ear can hear.

Section 4.6 Exercises

1. You go to the doctor and he gives you 13 milligrams of radioactive dye. After 12 minutes, 4.75 milligrams of dye remain in your system. To leave the doctor's office, you must pass through a radiation detector without sounding the alarm. If the detector will sound the alarm if more than 2 milligrams of the dye are in your system, how long will your visit to the doctor take, assuming you were given the dye as soon as you arrived?

2. You take 200 milligrams of a headache medicine, and after 4 hours, 120 milligrams remain in your system. If the effects of the medicine wear off when less than 80 milligrams remain, when will you need to take a second dose?

3. The half-life of Radium-226 is 1590 years. If a sample contains 200 mg, how many milligrams will remain after 1000 years?

4. The half-life of Fermium-253 is 3 days. If a sample contains 100 mg, how many milligrams will remain after 1 week?

5. The half-life of Erbium-165 is 10.4 hours. After 24 hours a sample has been reduced to a mass of 2 mg. What was the initial mass of the sample, and how much will remain after 3 days?

6. The half-life of Nobelium-259 is 58 minutes. After 3 hours a sample has been reduced to a mass of 10 mg. What was the initial mass of the sample, and how much will remain after 8 hours?

7. A scientist begins with 250 grams of a radioactive substance. After 225 minutes, the sample has decayed to 32 grams. Find the half-life of this substance.

8. A scientist begins with 20 grams of a radioactive substance. After 7 days, the sample has decayed to 17 grams. Find the half-life of this substance.

9. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (the half-life of carbon-14 is 5730 years)

10. A wooden artifact from an archeological dig contains 15 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (the half-life of carbon-14 is 5730 years)

11. A bacteria culture initially contains 1500 bacteria and doubles every half hour. Find the size of the population after: a) 2 hours, b) 100 minutes

12. A bacteria culture initially contains 2000 bacteria and doubles every half hour. Find the size of the population after: a) 3 hours, b) 80 minutes

13. The count of bacteria in a culture was 800 after 10 minutes and 1800 after 40 minutes.
    a. What was the initial size of the culture?
    b. Find the doubling period.
    c. Find the population after 105 minutes.
    d. When will the population reach 11000?

14. The count of bacteria in a culture was 600 after 20 minutes and 2000 after 35 minutes.
    a. What was the initial size of the culture?
    b. Find the doubling period.
    c. Find the population after 170 minutes.
    d. When will the population reach 12000?

15. Find the time required for an investment to double in value if invested in an account paying 3% compounded quarterly.

16. Find the time required for an investment to double in value if invested in an account paying 4% compounded monthly

17. The number of crystals that have formed after *t* hours is given by $n(t) = 20e^{0.013t}$. How long does it take the number of crystals to double?

18. The number of building permits in Pasco *t* years after 1992 roughly followed the equation $n(t) = 400e^{0.143t}$. What is the doubling time?

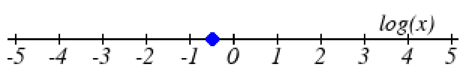
19. A turkey is pulled from the oven when the internal temperature is 165° Fahrenheit, and is allowed to cool in a 75° room. If the temperature of the turkey is 145° after half an hour,
    a. What will the temperature be after 50 minutes?
    b. How long will it take the turkey to cool to 110°?

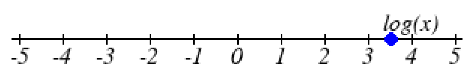
20. A cup of coffee is poured at 190° Fahrenheit, and is allowed to cool in a 70° room. If the temperature of the coffee is 170° after half an hour,
    a. What will the temperature be after 70 minutes?
    b. How long will it take the coffee to cool to 120°?

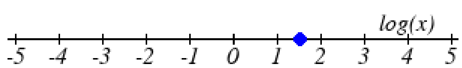
21. The population of fish in a farm-stocked lake after *t* years could be modeled by the equation $P(t) = \dfrac{1000}{1+9e^{-0.6t}}$.
    a. Sketch a graph of this equation
    b. What is the initial population of fish?
    c. What will the population be after 2 years?
    d. How long will it take for the population to reach 900?

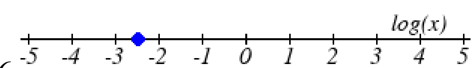
22. The number of people in a town that have heard rumor after *t* days can be modeled by the equation $N(t) = \dfrac{500}{1+49e^{-0.7t}}$.
    a. Sketch a graph of this equation
    b. How many people started the rumor?
    c. How many people have heard the rumor after 3 days??
    d. How long will it take 300 people to have heard the rumor?

Find the value of the number shown on each logarithmic scale

23. 

24. 

25. 

26. 

Plot each set of approximate values on a logarithmic scale

27. Intensity of sounds: Whisper: $10^{-10}\ W/m^2$, Vacuum: $10^{-4} W/m^2$, Jet: $10^{2}\ W/m^2$

28. Mass: Amoeba: $10^{-5} g$, Human: $10^{5} g$, Statue of Liberty: $10^{8} g$

29. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. At the same time there was an earthquake with magnitude 4.7 that caused only minor damage. How many times more intense was the San Francisco earthquake than the second one?

30. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. At the same time there was an earthquake with magnitude 6.5 that caused less damage. How many times more intense was the San Francisco earthquake than the second one?

31. One earthquake has magnitude 3.9. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake.

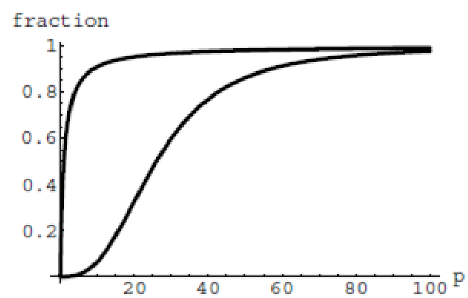
32. One earthquake has magnitude 4.8. If a second earthquake has 1200 times as much energy as the first, find the magnitude of the second quake.

33. A colony of yeast cells is estimated to contain 10^6 cells at time t = 0. After collecting experimental data in the lab, you decide that the total population of cells at time t hours is given by the function $f(t) = 10^6 e^{0.495105t}$ [UW]
    a. How many cells are present after one hour?
    b. How long does it take of the population to double?.
    c. Cherie, another member of your lab, looks at your notebook and says: ...that formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function. $f(t) = 10^6 (2.042727)^{0.693147t}$. Should you be worried by Cherie's remark?
    d. Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time *t*?

34. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function. [UW]
    a. If the intensity of light at the water's surface is I_0, find a formula for $I(d)$, the intensity of light at a depth of *d* meters. Your formula should depend on I_0 and *d*.
    b. At what depth will the light intensity be decreased to 1% of its surface intensity?

35. Myoglobin and hemoglobin are oxygen carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function $Y = M(p) = \frac{p}{1+p}$ calculates the fraction of myoglobin saturated with oxygen at a given pressure p torrs. For example, at a pressure of 1 torr, M(1) = 0.5, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function $Y = H(p) = \frac{p^{2.8}}{26^{2.8} + p^{2.8}}$ calculates the fraction of hemoglobin saturated with oxygen at a given pressure p. [UW]

a. The graphs of $M(p)$ and $H(p)$ are given here on the domain $0 \le p \le 100$; which is which?



b. If the pressure in the lungs is 100 torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?

c. The pressure in an active muscle is 20 torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

d. Define the efficiency of oxygen transport at a given pressure p to be $M(p) - H(p)$. What is the oxygen transport efficiency at 20 torrs? At 40 torrs? At 60 torrs? Sketch the graph of $M(p) - H(p)$; are there conditions under which transport efficiency is maximized (explain)?

36. The length of some fish are modeled by a von Bertalanffy growth function. For Pacific halibut, this function has the form $L(t) = 200\left(1 - 0.957e^{-0.18t}\right)$ where $L(t)$ is the length (in centimeters) of a fish t years old. [UW]

a. What is the length of a new-born halibut at birth?

b. Use the formula to estimate the length of a 6–year–old halibut.

c. At what age would you expect the halibut to be 120 cm long?

d. What is the practical (physical) significance of the number 200 in the formula for $L(t)$?

37. A cancerous cell lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its mitotic cell cycle (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model. [UW]
    a. Find a formula $C(t)$ for the number of cancerous skin cells after *t* hours.
    b. Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after *t* hours. When will the volume of cancerous cells be 1 cm^3?

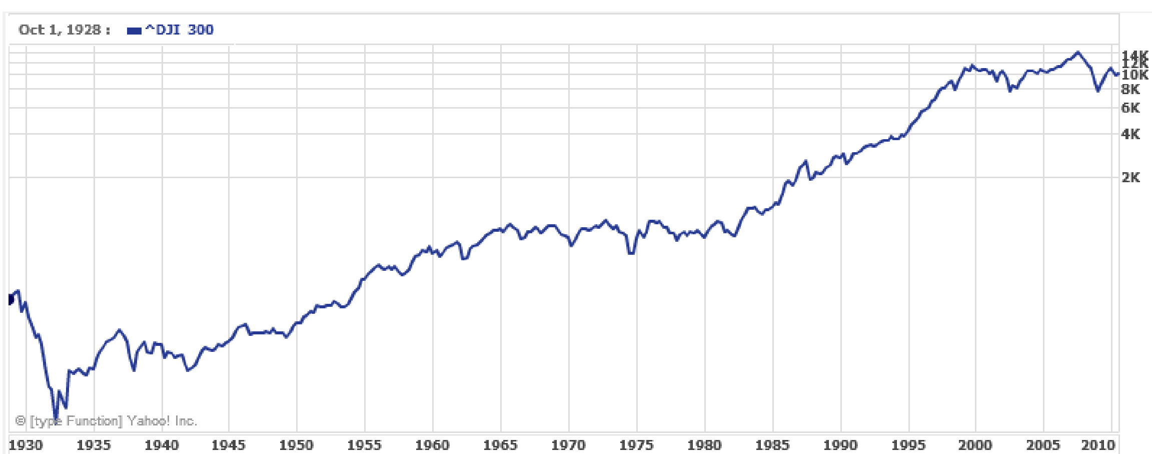
38. A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time. [UW]
    a. How long did it take the population to double?
    b. How long did it take the population to triple?
    c. When were there be 10,000 ants on board?
    d. There also was an exponentially-growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

39. The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders. How long (in days) does it take the population of spiders to triple? [UW]

Section 4.7 Fitting Exponentials to Data

In the previous section, we saw numbers lines using logarithmic scales. It is also common to see two dimensional graphs with one or both axes represented on a logarithmic scale.

One common use of a logarithmic scale on the vertical axis is in graphing quantities that are changing exponentially, since it helps reveal relative differences. This is commonly used in stock charts, since values historically have grown exponentially over time. Both stock charts below show the Dow Jones Industrial Average, from 1928 to 2010.



Both charts have a linear horizontal scale, but the first graph has a linear vertical scale, while the second has a logarithmic vertical scale. The first scale is the one we are more used to, and shows what appears to be a strong exponential trend, at least up until the year 2000.

Example 1

There were stock market drops in 1929 and 2008. Which was larger?

In the first graph, the stock market drop around 2008 looks very large, and in terms of dollar values, it was indeed a large drop. However the second graph shows relative changes, and the drop in 2009 seems less major on this graph, and in fact the drop starting in 1929 was, percentage-wise, much more significant.

Specifically, in 2008, the Dow value dropped from about 14,000 to 8,000, a drop of 6,000. This is obviously a large value drop, and accounts to about a 43% drop. In 1929, the Dow value dropped from a high of around 380 to a low of 42 by July of 1932. While value-wise this drop of 338 is smaller than the 2008 drop, but this corresponds to a 89% drop, a much larger relative drop than in 2008. The logarithmic scale shows these relative changes.

The second graph above, in which one axis uses a linear scale and the other axis uses a logarithmic scale, is an example of a **semi-log** graph.

Semi-log and Log-log Graphs

A **semi-log** graph is a graph with one axis using a linear scale and one axis using a logarithmic scale.
A **log-log** graph is a graph with both axes using logarithmic scales.

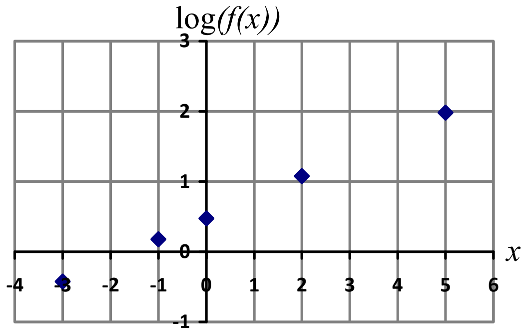
Example 2

Plot 5 points from the equation $f(x) = 3(2)^x$ on a semi-log graph with a logarithmic scale on the vertical axis.

To do this, we need to find 5 points on the graph, then calculate the logarithm of the output value. Arbitrarily choosing 5 input values,

x	*f(x)*	*log(f(x))*
-3	$3(2)^{-3} = \frac{3}{8}$	-0.426
-1	$3(2)^{-1} = \frac{3}{2}$	0.176
0	$3(2)^{0} = 3$	0.477
2	$3(2)^{2} = 12$	1.079
5	$3(2)^{5} = 96$	1.982

Plotting these values on a semi-log graph,



Notice that on this semi-log scale, values from the exponential function appear linear. We can show this is expected by utilizing logarithmic properties. For the function $f(x) = ab^x$, finding log(*f(x)*) gives

$\log(f(x)) = \log(ab^x)$ Utilizing the sum property of logs,

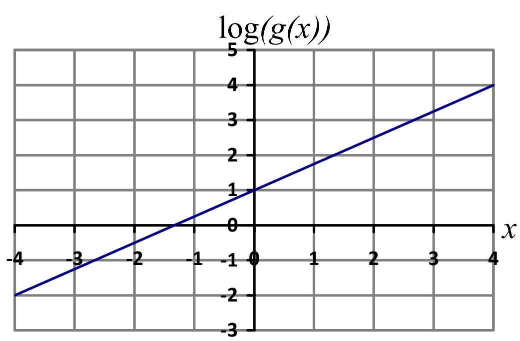
$\log(f(x)) = \log(a) + \log(b^x)$ Now utilizing the exponent property,

$\log(f(x)) = \log(a) + x\log(b)$

This relationship is linear, with log(*a*) as the vertical intercept, and log(*b*) as the slope. This relationship can also be utilized in reverse.

Example 3

An exponential graph is plotted on a semi-log graph below. Find an equation for the exponential function *g(x)* that generated this graph.



The graph is linear, with vertical intercept at (0,1). Looking at the change between the points (0,1) and (4,4), we can determine the slope of the line is $\frac{3}{4}$. Since the output is log(*g(x)*), this leads to the equation $\log(g(x)) = 1 + \frac{3}{4}x$.

We can solve this formula for *g(x)* by rewriting as an exponential and simplifying:

$\log(g(x)) = 1 + \frac{3}{4}x$ Rewriting as an exponential,

$g(x) = 10^{1+\frac{3}{4}x}$ Breaking this apart using exponent rules,

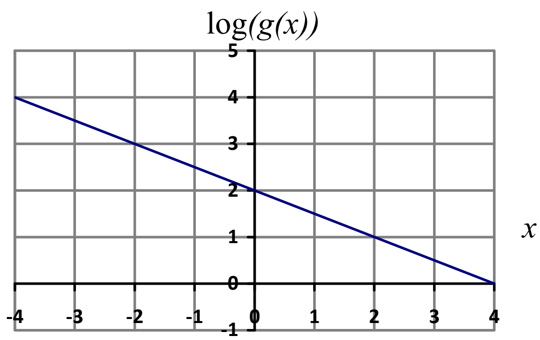
$g(x) = 10^1 \cdot 10^{\frac{3}{4}x}$ Using exponent rules to group the second factor,

$g(x) = 10^1 \cdot \left(10^{\frac{3}{4}}\right)^x$ Evaluating the powers of 10,

$g(x) = 10(5.623)^x$

Try it Now

1. An exponential graph is plotted on a semi-log graph below. Find an equation for the exponential function *g(x)* that generated this graph.



Fitting Exponential Functions to Data

Some technology options provide dedicated functions for finding exponential functions that fit data, but many only provide functions for fitting linear functions to data. The semi-log scale provides us with a method to fit an exponential function to data by building upon the techniques we have for fitting linear functions to data.

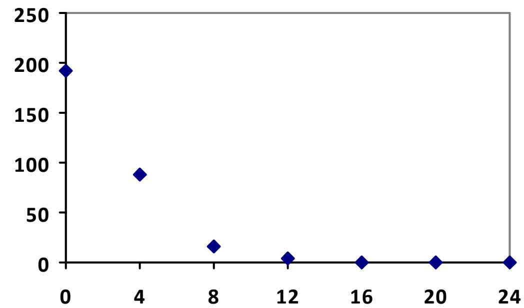
To fit an exponential function to a set of data using linearization

1. Find the log of the data output values
2. Find the linear equation that fits the (input, log(output)) pairs. This equation will be of the form log(*f(x)*) = *b* + *mx*
3. Solve this equation for the exponential function *f(x)*

Example 4

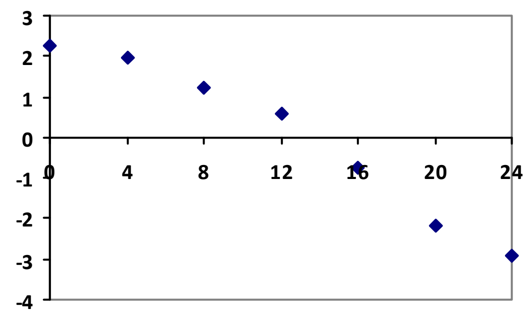
The table below shows the cost in dollars per megabyte of storage space on computer hard drives from 1980 to 2004[5], and the data is shown on a standard graph to the right, with the input changed to years after 1980

Year	Cost per MB
1980	192.31
1984	87.86
1988	15.98
1992	4
1996	0.173
2000	0.006849
2004	0.001149



This data appears to be decreasing exponentially. To find an equation for this decay, we would start by finding the log of the costs.

Year	Cost per MB	log(Cost)
1980	192.31	2.284002
1984	87.86	1.943791
1988	15.98	1.203577
1992	4	0.60206
1996	0.173	-0.76195
2000	0.006849	-2.16437
2004	0.001149	-2.93952



As expected, the graph of the log of costs appears fairly linear, suggesting the original data will be fit reasonably well with an exponential equation. Using technology, we can find an equation to fit the log(Cost) values. Using *t* as years after 1980, regression gives the equation:

$$\log(C(t)) = 2.794 - 0.231t$$

Solving for *C(t)*,

$$C(t) = 10^{2.794 - 0.231t}$$

$$C(t) = 10^{2.794} \cdot 10^{-0.231t}$$

$$C(t) = 10^{2.794} \cdot \left(10^{-0.231}\right)^t$$

$$C(t) = 622 \cdot (0.5877)^t$$

This equation suggests that the cost per megabyte for storage on computer hard drives is decreasing by about 41% each year.

[5] Selected values from http://www.swivel.com/workbooks/26190-Cost-Per-Megabyte-of-Hard-Drive-Space, retrieved Aug 26, 2010

Using this function, we could predict the cost of storage in the future. Predicting the cost in the year 2020 ($t = 40$):
$C(40) = 622 \cdot (0.5877)^{40} \approx 0.000000364$ dollars per megabyte, a really small number. That is equivalent to $0.36 per terabyte of hard drive storage.

Comparing the values predicted by this model to the actual data, we see the model matches the original data in order of magnitude, but the specific values appear quite different. This is, unfortunately, the best exponential that can fit the data. It is possible that a different model would fit the data better, or there could just be a wide enough variability in the data that no relatively simple model would fit the data any better.

Year	Actual Cost per MB	Cost predicted by model
1980	192.31	622.3
1984	87.86	74.3
1988	15.98	8.9
1992	4	1.1
1996	0.173	0.13
2000	0.006849	0.015
2004	0.001149	0.0018

Try it Now

2. The table below shows the value V, in billions of dollars, of US imports from China t years after 2000.

year	2000	2001	2002	2003	2004	2005
t	0	1	2	3	4	5
V	100	102.3	125.2	152.4	196.7	243.5

This data appears to be growing exponentially. Linearize this data and build a model to predict how many billions of dollars of imports we could expect in 2011.

Important Topics of this Section

Semi-log graph
Log-log graph
Linearizing exponential functions
Fitting an exponential equation to data

Try it Now Answers

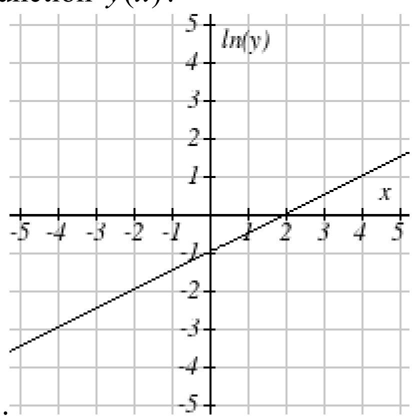
1. $f(x) = 100(0.3162)^x$
2. $V(t) = 90.545(1.2078)^t$. Predicting in 2011, $V(11) = 722.45$ billion dollars

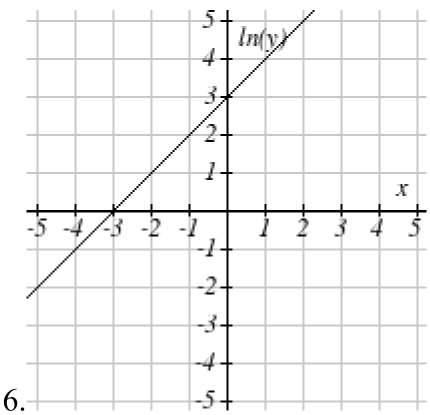
Section 4.7 Exercises

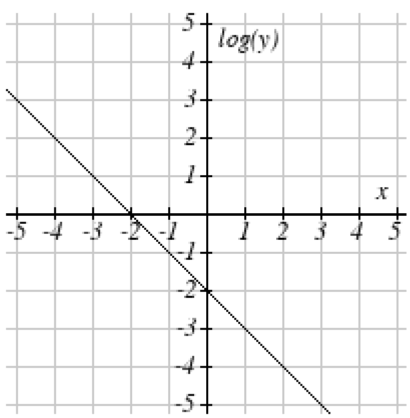
Graph each function on a semi-log scale, the find a formula for the linearized function in the form $\log(f(x)) = mx + b$

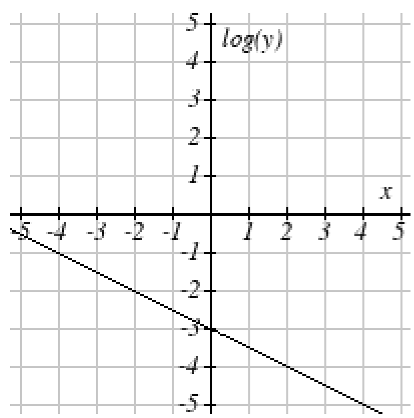
1. $f(x) = 4(1.3)^x$
2. $f(x) = 2(1.5)^x$
3. $f(x) = 10(0.2)^x$
4. $f(x) = 30(0.7)^x$

The graph below is on a semi-log scale, as indicated. Find an equation for the exponential function $y(x)$.

5. 

6. 

7. 

8. 

Use regression to find an exponential equation that best fits the data given.

9.

x	1	2	3	4	5	6
y	1125	1495	2310	3294	4650	6361

10.

x	1	2	3	4	5	6
y	643	829	920	1073	1330	1631

11.

x	1	2	3	4	5	6
y	555	383	307	210	158	122

12.

x	1	2	3	4	5	6
y	699	701	695	668	683	712

13. Total expenditures (in billions of dollars) in the US for nursing home care are shown below. Use regression to find an exponential equation that models the data. What does the model predict expenditures will be in 2015?

Year	1990	1995	2000	2003	2005	2008
Expenditure	53	74	95	110	121	138

14. Light intensity as it passes through decreases exponentially with depth. The data below shows the light intensity (in lumens) at various depths. Use regression to find an equation that models the data. What does the model predict the intensity will be at 25 feet?

Depth (ft)	3	6	9	12	15	18
Lumen	11.5	8.6	6.7	5.2	3.8	2.9

15. The average price of electricity (in cents per kilowatt hour) from 1990-2008 is given below. Determine if a linear or exponential model better fits the data, and use the better model to predict the price of electricity in 2014.

Year	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Cost	7.83	8.21	8.38	8.36	8.26	8.24	8.44	8.95	10.40	11.26

16. The average cost of a loaf of white bread from 1986-2008 is given below. Determine if a linear or exponential model better fits the data, and use the better model to predict the price of a loaf of bread in 2016.

Year	1986	1988	1990	1995	1997	2000	2002	2004	2006	2008
Cost	0.57	0.66	0.70	0.84	0.88	0.99	1.03	0.97	1.14	1.42

Solutions to Selected Exercises

Chapter 1

Section 1.1

1\. a. $f(40) = 13$

b. 2 Tons of garbage per week is produced by a city with a population of 5,000.

3\. a. In 1995 there are 30 ducks in the lake

b. In 2000 there are 40 ducks in the late

5\. a ,b, d, e 7. a, b 9. a, b, d

11\. b 13. b, c, e, f 15. $f(1) = 1, \quad f(3) = 1$

17\. $g(2) = 4, \quad g(-3) = 2$ 19. $f(3) = 53, \quad f(2) = 1$

	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
21.	8	6	4	2	0
23.	49	18	34	4	21
25.	4	-1	0	1	-4
27.	4	4.414	4.732	5	5.236
29.	-4	-6	-6	-4	0
31.	5	DNE	-3	-1	-1/3
33.	1/4	1/2	1	2	4

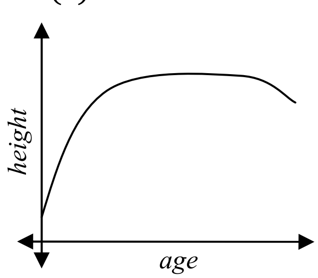
35\. a. -6 b.-16 37. a. 5 b. $-\frac{5}{3}$

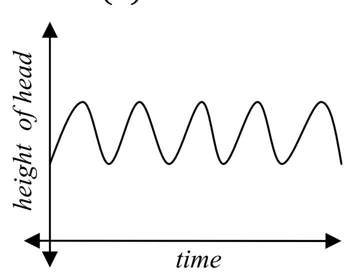
39\. a. iii b. viii c. I d. ii e. vi f. iv g. v h. vii

41\. a. iv b. ii c. v d. I e. vi f. iii

43\. $(x-3)^2 + (y+9)^2 = 36$

45\. (a) (b)





47a. *t* b. *a* c. *r* d. L: (c, t) and K: (a, p)

Section 1.2

1. D: [-5, 3) R: [0,2]

3. D: $2 < t \le 8$ R: $6 \le g(t) < 8$

5. D: [0,4] R: [-3, 0]

7. $[2,\infty)$

9. $(-\infty,3]$

11. $(-\infty,6)\cup(6,\infty)$

13. $(-\infty,-\frac{1}{2})\cup\left(-\frac{1}{2},\infty\right)$

15. $[-4,4)\cup(4,\infty)$

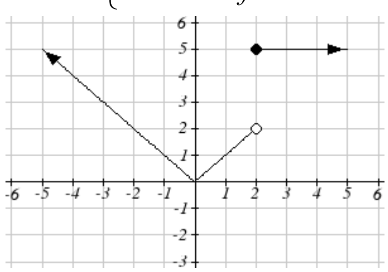
17. $(-\infty,-11)\cup(-11,2)\cup(2,\infty)$

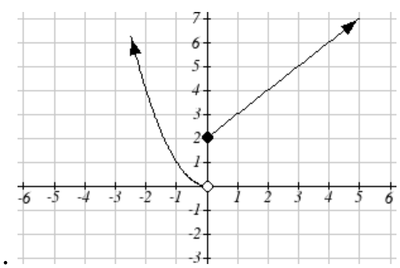
	$f(-1)$	$f(0)$	$f(2)$	$f(4)$
19.	-4	6	20	34
21.	-1	-2	7	5
23.	-5	3	3	16

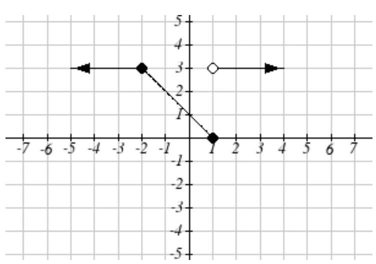
25. $f(x)=\begin{cases} 2 & if \quad -6 \le x \le -1 \\ -2 & if \quad -1 < x \le 2 \\ -4 & if \quad 2 < x \le 4 \end{cases}$

27. $f(x)=\begin{cases} 3 & if \quad x \le 0 \\ x^2 & if \quad x > 0 \end{cases}$

29. $f(x)=\begin{cases} 2x+3 & if \quad -3 \le x < -1 \\ x-1 & if \quad -1 \le x \le 2 \\ -2 & if \quad 2 < x \le 5 \end{cases}$

31. 

33. 

35. 

Section 1.3

1. a) 6 million dollars per year b) 2 million dollars per year

3. $\frac{4-5}{4-1}=-\frac{1}{3}$

5. 6

7. 27

9. $\frac{352}{27}$

11. $4b+4$

13. 3

15. $-\frac{1}{13h+169}$

17. $9+9h+3h^2$

19. $4x+2h$

21. Increasing: $(-1.5,2)$. Decreasing: $(-\infty,-1.5)\cup(2,\infty)$

23. Increasing: $(-\infty,1)\cup(3,4)$. Decreasing: $(1,3)\cup(4,\infty)$

25. Increasing, concave up

27. Decreasing, concave down

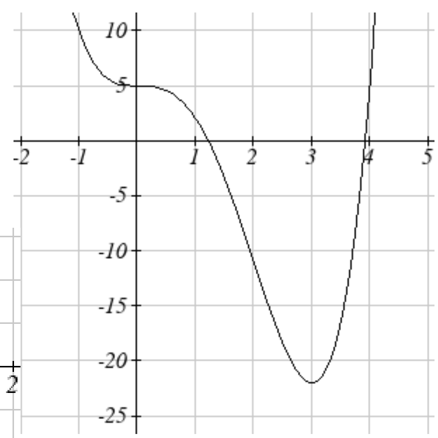
29. Decreasing, concave up

31. Increasing, concave down

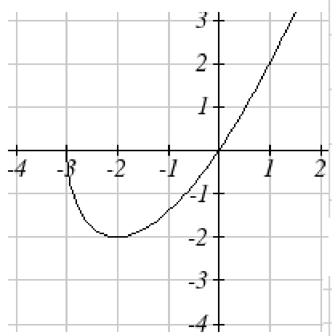
33. Concave up $(-\infty,1)$. Concave down $(1,\infty)$. Inflection point at (1, 2)

35. Concave down $(-\infty,3)\cup(3,\infty)$

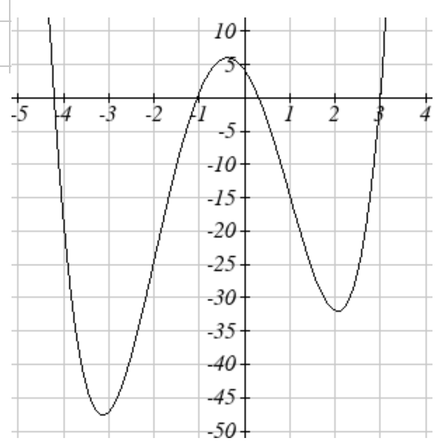
37. Local minimum at (3, -22). Inflection at (2, -11).
Increasing on $(3,\infty)$. Decreasing $(-\infty,3)$
Concave up $(-\infty,0)\cup(2,\infty)$. Concave down $(0,2)$



39. Local minimum at (-2, -2)
Decreasing $(-3,-2)$
Increasing $(-2,\infty)$
Concave up $(-3,\infty)$



41. Local minimums at (-3.152, -47.626)
and (2.041, -32.041)
Local maximum at (-0.389, 5.979)
Inflection points at (-2, -24) and (1, -15)
Increasing $(-3.152,-0.389)\cup(2.041,\infty)$
Decreasing $(-\infty,-3.152)\cup(-0.389,2.041)$
Concave up $(-\infty,-2)\cup(1,\infty)$
Concave down $(-2,1)$



Section 1.4

1. $f(g(0)) = 36$. $g(f(0)) = -57$

3. $f(g(0)) = 4$. $g(f(0)) = 4$

5. 4 7. 9 9. 4 11. 7 13. 0 15. 4 17. 3 19. 2

21. $f(g(x)) = \frac{x}{7}$ $g(f(x)) = 7x - 36$

23. $f(g(x)) = x + 3$ $g(f(x)) = \sqrt{x^2 + 3}$

25. $f(g(x)) = |5x + 1|$ $g(f(x)) = 5|x| + 1$

27. $f(g(h(x))) = \left(\sqrt{x} - 6\right)^4 + 6$

29a. $(0,2) \cup (2,\infty)$ b. $(-\infty,2) \cup (2,\infty)$ c. $(0,\infty)$

31. b 33a. $r(V(t)) = \sqrt[3]{\frac{3(10+20t)}{4\pi}}$ b. 208.94

35. $g(x) = x + 2, f(x) = x^2$ 37. $f(x) = \frac{3}{x}, g(x) = x - 5$

39. $f(x) = 3 + \sqrt{x}, g(x) = x - 2$, or $f(x) = 3 + x, g(x) = \sqrt{x - 2}$

41a. $f(f(x)) = a(ax + b) + b = (a^2)x + (ab + b)$

b. $g(x) = \sqrt{6}x - \frac{8}{\sqrt{6} + 1}$ or $g(x) = -\sqrt{6}x - \frac{8}{1 - \sqrt{6}}$

43a. $C(f(s)) = \dfrac{70\left(\frac{s}{60}\right)^2}{10 + \left(\frac{s}{60}\right)^2}$ b. $C(g(h)) = \dfrac{70(60h)^2}{10 + (60h)^2}$

c. $v(C(m)) = \frac{5280}{3600}\left(\frac{70m^2}{10 + m^2}\right)$

Section 1.5

1. Horizontal shift right 49 units 3. Horizontal shift left 3 units

5. Vertical shift up 5 units

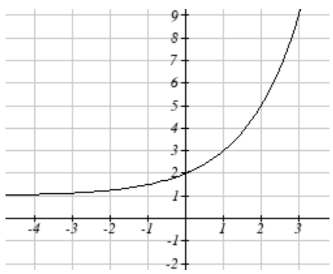
7. Vertical shift down 2 units

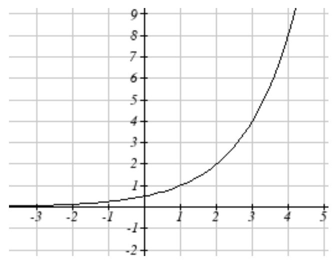
9. Horizontal shift right 2 units, Vertical shift up 3 units

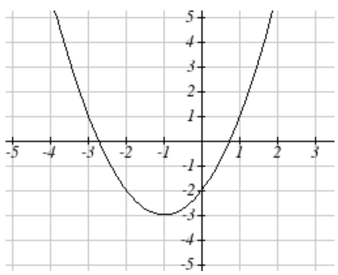
11. $f(x+2)+1=\sqrt{x+2}+1$

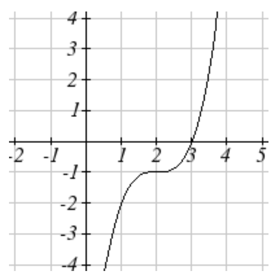
13. $f(x-3)-4=\dfrac{1}{x-3}-4$

15. $g(x)=f(x-1)$, $\quad h(x)=f(x)+1$

17. 

19. 

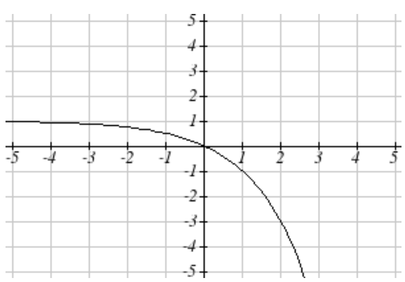
21. 

23. 

25. $y=|x-3|-2$

27. $y=\sqrt{x+3}-1$

29. $y=-\sqrt{x}$

31. 

33a. $-f(-x)=-6^{-x}$

b. $-f(x+2)-3=-6^{x+2}-3$

35. $y=-(x+1)^2+2$

37. $y=\sqrt{-x}+1$

39a. Even b. Neither c. Odd

41. Reflect *f(x)* about the *x*-axis

43. Vertically stretch *y* values by 4

45. Horizontally compress *x* values by 1/5

47. Horizontally stretch *x* values by 3

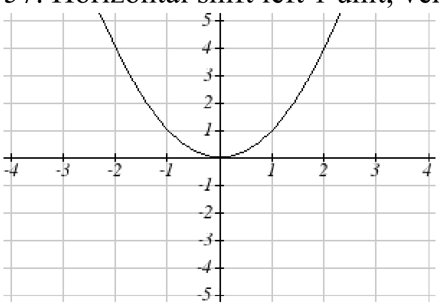
49. Reflect *f(x)* about the *y*-axis and vertically stretch *y* values by 3

51. $f(-4x) = |-4x|$

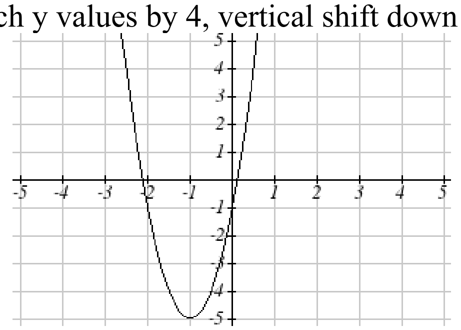
53. $\frac{1}{3}f(x+2)-3 = \frac{1}{3(x+2)^2}-3$

55. $f(2(x-5))+1 = (2(x-5))^2+1$

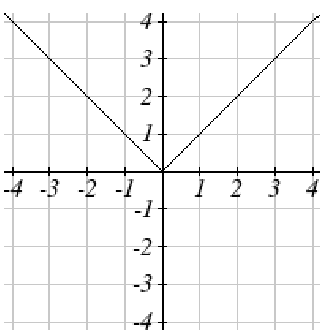
57. Horizontal shift left 1 unit, vertical stretch y values by 4, vertical shift down 5 units



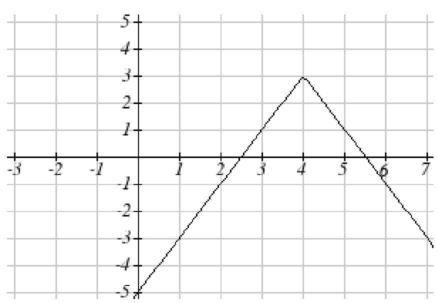
becomes



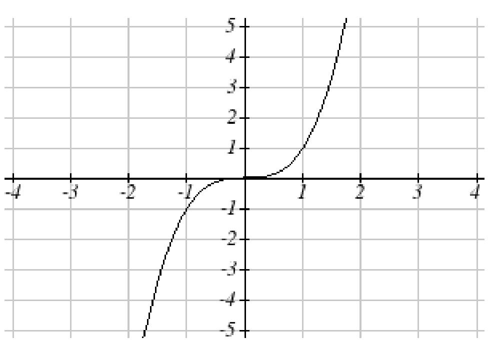
59. Horizontal shift right 4 units, vertical stretch y values by 2, reflect over x axis, vertically shift up 3 units.



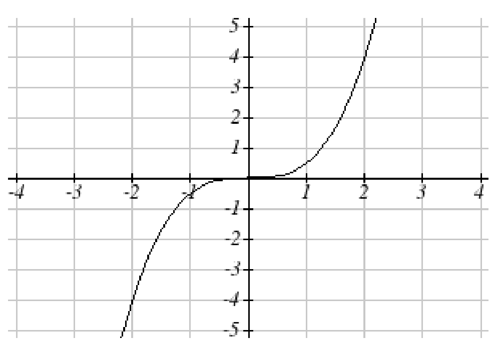
becomes



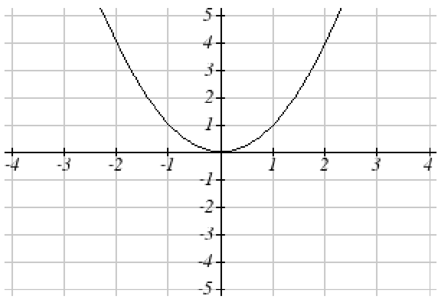
61. Vertically compress y values by ½



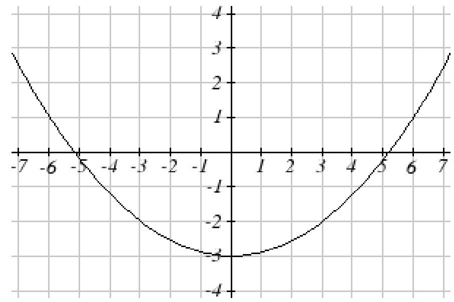
becomes



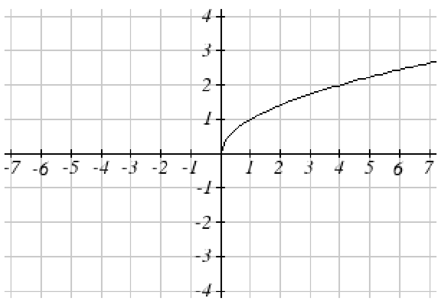
63. Horizontally stretch x values by 3, vertical shift down 3 units



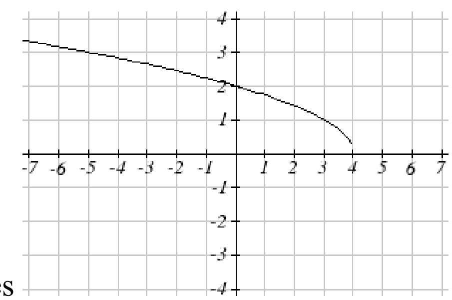
becomes



65. Reflected over the y axis, horizontally shift right 4 units $a(x)=\sqrt{-(x-4)}$



becomes



67. This function is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$

69. This function is decreasing on $(-\infty,4)$

71. This function is concave down on $(-3,\infty)$ and concave up on $(-\infty,-3)$

73. This function is concave up everywhere

75. $f(-x)$

77. $3f(x)$

79. $2f(-x)$

81. $2f\left(\frac{1}{2}x\right)$

83. $2f(x)-2$

85. $-f(x+1)+3$

87. $y=-2(x+2)^2+3$

89. $y=\left(\frac{1}{2}(x-1)\right)^3+2$

91. $y=\sqrt{2(x+2)}+1$

93. $y=\frac{-1}{(x-2)^2}+3$

95. $y=-2|x+1|+3$

97. $y=\sqrt[3]{-\frac{1}{2}(x-2)}+1$

99a. $Domain: 3.5 \le x \le 6$

d. $Range: -9 \le y \le 7$

Section 1.6

1. 6 3. -4 5. ½

7a. 3 b. 2 c. 2 d. 3

9a. 0 b. 7 c. 1 d. 3

11.

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

13. $f^{-1}(x) = x - 3$ 15. $f^{-1}(x) = -x + 2$ 17. $f^{-1}(x) = \dfrac{x-7}{11}$

19. Restricted domain $x \geq -7, f^{-1}(x) = \sqrt{x} - 7$

21. Restricted domain $x \geq 0,\ f^{-1}(x) = \sqrt{x+5}$

23a. $f(g(x)) = \left(\sqrt[3]{x+5}\right)^3 - 5 = x$ b. $g(f(x)) = \sqrt[3]{x^3 - 5 + 5} = x$

c. This means that they are inverse functions (of each other)

Chapter 2

Section 2.1

1. $P(t) = 1700t + 45000$ 3. $D(t) = 10 + 2t$ 5. $M(n) = 4 - 2n$

7. Increasing 9. Decreasing 11. Decreasing

13. Increasing 15. Decreasing 17. 3

19. $-\dfrac{1}{3}$ 21. $\dfrac{4}{5}$ 23. $\dfrac{2}{3}$

25. -0.05 mph (or 0.05 miles per hour toward her home)

27. Population is decreasing by 400 people per year

29. Monthly charge in dollars has an initial base charge of \$24, and increases by \$0.10 for each minute talked

31. Terry started at an elevation of 3,000 ft and is descending by 70ft per second.

33. $y = \dfrac{3}{5}x - 1$ 35. $y = 3x - 2$ 37. $y = -\dfrac{1}{3}x + \dfrac{11}{3}$

39. $y = -1.5x - 3$ 41. $y = \frac{2}{3}x + 1$ 43. $y = -2x + 3$

45. $P(n) = -0.004n + 34$

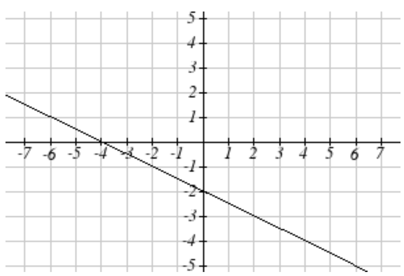
47. The 1st ,3rd & 4th tables are linear: respectively

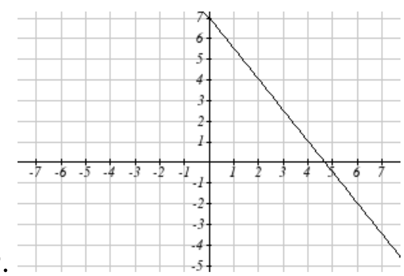
1. $g(x) = -3x + 5$ 3. $f(x) = 5x - 5$ 4. $k(x) = 3x - 2$

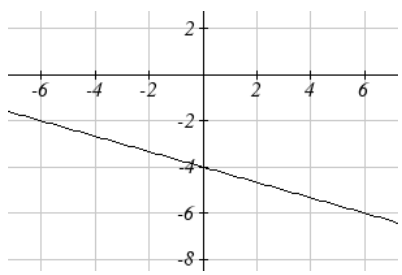
49a. $C = \frac{5}{9}F - \frac{160}{9}$ b. $F = \frac{9}{5}C + 32$ c. $-9.4°F$

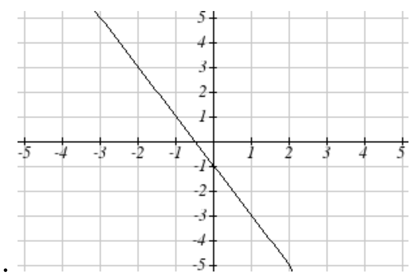
Section 2.2

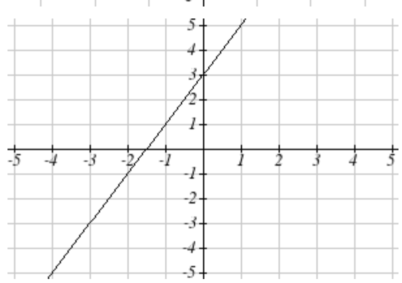
1. E 3. D 5. B

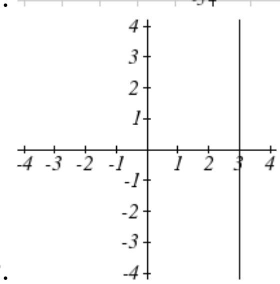
7. 

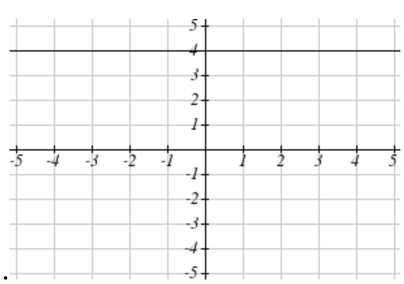
9. 

11. 

13. 

17. 

19. 

21. 

23. a. $g(x) = \frac{3}{4}(x + 2) - 4$ b. ¾ c. -5/2

25. $y = 3$

27. $x = -3$

	Vertical Intercept	Horizontal Intercept
29.	(0,2)	(2,0)
31.	(0,-5)	(5/3, 0)
33.	(0,4)	(-10,0)

35. Line 1: $m = -10$ Line 2: $m = -10$ Parallel

37. Line 1: $m = -2$ Line 2: $m = 1$ Neither

39. Line 1: $m = -\frac{2}{3}$ Line 2: $m = \frac{3}{2}$ Perpendicular

41. $y = -5x - 2$ 43. $y = \frac{1}{2}t + 1$ 45. (-1,1)

47. (1.2, 10) 49. Plan B saves money if the miles are $> 111\frac{1}{9}$

Section 2.3

1a. 696 people b. 4 years c. 174 people per year

d. 305 people e. $P(t) = 305 + 174t$ f. 2219 people.

3a. $C(x) = 0.35x + 30$

b. The flat monthly fee is $10 and there is an additional $0.15 fee for each additional minute used

c. $113.05

5a. $P(t) = 190t + 4170$ b. 6640 moose

7a. $R(t) = 16 - 2.1t$ b. 5.5 billion cubic feet c. During the year 2017

9. More than 133 minutes 11. More than $\$42,857.14$ worth of jewelry

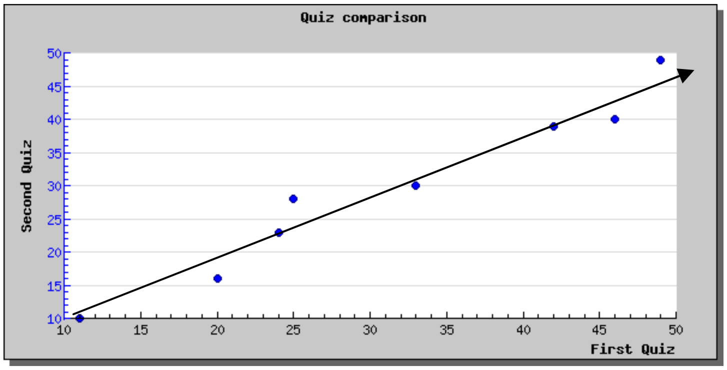
13. 20.012 square units 15. 6 square units

17. $A = -\frac{b^2}{2m}$

19a. Hawaii b. $80,640 c. During the year 1933

21. 26.225 miles

Section 2.4

1. 

http://www.mathcracker.com/scatter_plot.php

3. $y = 1.971x - 3.519, \quad r = 0.967$

5. $y = -0.901x + 26.04, \quad r = -0.968$

7. $17.483 \approx 17\, situps$

9. D

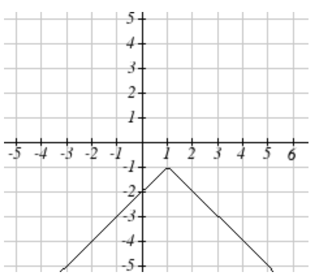
11. A

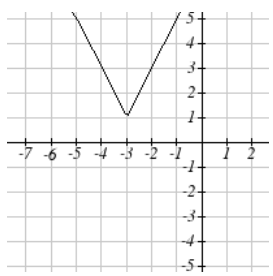
13. Yes, trend appears linear because r =0.994 and will exceed 35% near the end of the year 2019.

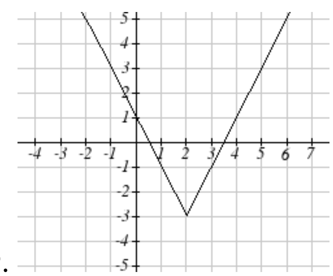
Section 2.5

1. $y = \frac{1}{2}|x+2|+1$

3. $y = -3|x-3|+3$

5. 

7. 

9. 

11. $x = -\frac{9}{5} \quad or \quad x = \frac{13}{5}$

13. $x = \frac{1}{2} \quad or \quad x = \frac{15}{2}$

15. $x = -\frac{5}{3} \quad or \quad x = -\frac{1}{3}$

	Horizontal Intercepts	Vertical Intercept
17.	(-6, 0) and (4, 0)	(0, -8)
19.	none	(0, -7)

21. $-11 < x < 1$ or $(-11, 1)$

23. $x \geq 5$, $x \leq -1$ or $(-\infty, -1] \cup [5, \infty)$

25. $-\frac{13}{3} < x < -\frac{5}{3}$ or $(-\frac{13}{3}, -\frac{5}{3})$

Chapter 3

Section 3.1

1. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to \infty$
3. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to -\infty$
5. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to -\infty$
7. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to \infty$
9. 7^{th} Degree, Leading coefficient 4
11. 2^{nd} Degree, Leading coefficient -1
13. 4^{th} Degree, Leading coefficient -2
15. 3^{rd} Degree, Leading coefficient 6
17. As $x \to \infty,\ f(x) \to -\infty$ As $x \to -\infty,\ f(x) \to -\infty$
19. As $x \to \infty,\ f(x) \to \infty$ As $x \to -\infty,\ f(x) \to \infty$
21. intercepts: 5, turning points: 4 23. 3
25. 5 27. 3 29. 5
31. Horizontal Intercepts (1,0), (-2, 0), (3, 0) Vertical Intercept (0, 12)
33. Horizontal Intercepts (1/3, 0) (-1/2, 0) Vertical Intercept (0, 2)

Section 3.2

1. $f(x) = (x-2)^2 - 3$ 3. $f(x) = -2(x-2)^2 + 7$ 5. $f(x) = \frac{1}{2}(x-3)^2 - 1$

	Vertex	Vertical Intercept	Horizontal Intercepts
7.	$(-2.5, -0.5)$	(0,12)	(-2, 0) (-3, 0)
9.	$(2.5, -8.5)$	(0,4)	(0.438, 0) (4.562,0)
11.	$(0.75, 1.25)$	(0,-1)	(0.191, 0) (1.309, 0)

13. $f(x) = (x-6)^2 - 4$ 15. $f(x) = 2(x+2)^2 - 18$ 17. b = 32 and c = -25
19. $f(x) = -\frac{2}{3}(x+3)(x-1)$ 21. $f(x) = \frac{3}{5}(x-2)(x-5)$
23. $f(x) = -\frac{1}{4}(x-4)^2$ 25. $f(x) = -\frac{1}{9}(x+3)^2 + 2$
27a. 234m b. 2909.561 ft c. 47.735 seconds
29a. 3 ft b. 111 ft c. 72.497 ft
31. 24.91 in by 24.91 in
33. 125 ft by $83\frac{1}{3}$ ft
35. 24.6344 cm
37. $10.70

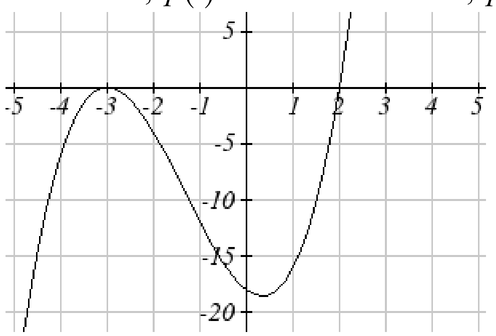
Section 3.3

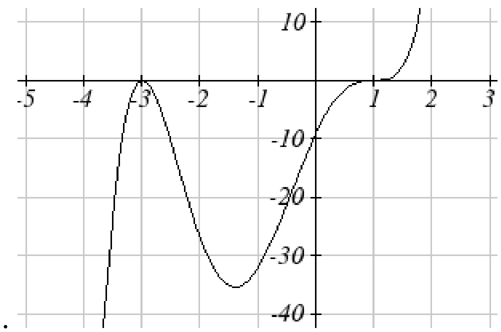
C(t)	*C*, intercepts	*t*, intercepts
1.	(0,48)	(4,0), (-1,0), (6,0)
3.	(0,0)	(0,0), (2,0), (-1,0)
5.	(0,0)	(0,0), (1,0), (3,0)

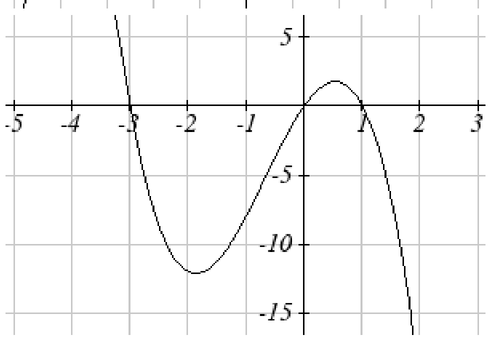
7. (-1.646, 0) (3.646, 0) (5,0)

9. As $t \to \infty,\ h(t) \to \infty \qquad t \to -\infty,\ h(t) \to -\infty$

11. As $t \to \infty,\ p(t) \to -\infty \qquad t \to -\infty,\ p(t) \to -\infty$

13. 

15. 

17. 

19. $(3, \infty)$

21. $(-\infty, -2) \cup (1, 3)$

23. $[3.5, 6]$

25. $(-\infty, 1] \cup (4, \infty)$

27. $[3, \infty)$

29. $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$

31. $y = -\frac{2}{3}(x+2)(x-1)(x-3)$

33. $y = \frac{1}{3}(x-1)^2(x-3)^2(x+3)$

35. $y = -15(x-1)^2(x-3)^3$

37. $y = \frac{1}{2}(x+2)(x-1)(x-3)$

39. $y = -(x+1)^2(x-2)$

41. $y = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

43. $y = \frac{1}{24}(x+4)(x+2)(x-3)^2$

45. $y = \frac{1}{12}(x+2)^2(x-3)^2$

47. $y = \frac{1}{6}(x+3)(x+2)(x-1)^3$

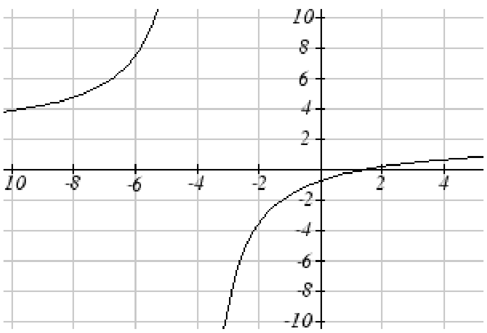
49. $y = -\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

51. Base 2.58, Height 3.336

Section 3.4

1. D 3. A

	Vertical Asymptotes	Horizontal Asymptote	Vertical *y*-Intercept	Horizontal *x*-intercept
5.	$x=-4$	$y=2$	(0,-3/4)	(3/2, 0)
7.	$x=2$	$y=0$	(0,1)	DNE
9.	$x=-4, 1\frac{1}{3}$	$y=1$	(0, 5/16)	(-1/3, 0), (5,0)
11.	$x=-1$, hole at $x=1$	$y=1$	(0,3)	(-3, 0)
13.	$x=4$	none y=2x (oblique)	(0, ¼)	(-1, 0), (1/2, 0)
15.	$x=0,\ 4$	$y=0$	DNE	(-2, 0), (2/3, 0)
17.	$x=-2,\ 4$	$y=1$	(0, -15/16)	(1, 0), (-3, 0), (5, 0)

5.


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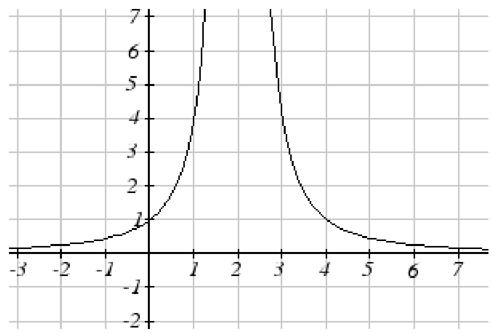

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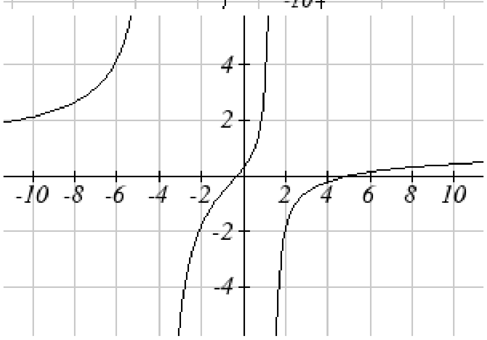

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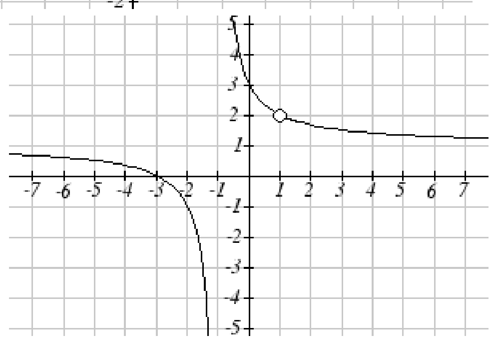

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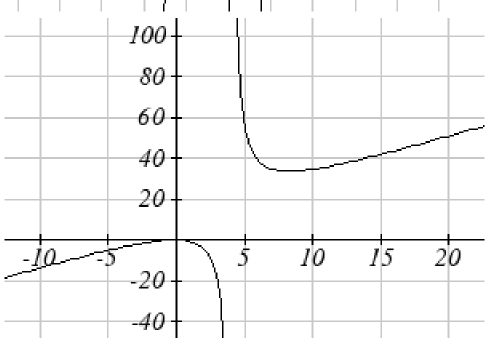

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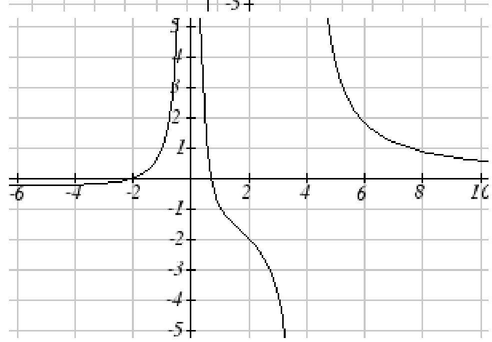

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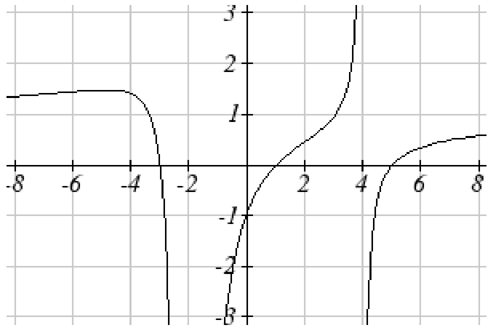
17. 

19. $y = \dfrac{50(x-2)(x+1)}{(x+5)(x-5)}$

21. $y = \dfrac{7(x-4)(x+6)}{(x+4)(x+5)}$

23. $y = \dfrac{1(x-2)^2}{2(x+1)}$

25. $y = \dfrac{4(x-3)}{(x+3)(x-4)}$

27. $y = \dfrac{27(x-2)}{(x+3)(x-3)^2}$

29. $y = \dfrac{1(x+3)(x-2)}{3(x-1)}$

31. $y = \dfrac{6(x-1)^2}{(x+3)(x-2)^2}$

33. $y = -\dfrac{2(x)(x-3)}{(x+3)(x-4)}$

35. $y = \dfrac{2(x-1)^3}{(x+1)(x-2)^2}$

37. $y = \dfrac{(x-4)(x-2)}{(x-4)(x+1)}$

39. a. $C(n) = \dfrac{4}{20+n}$ b. $C(10) \approx 13.33\%$ c. 80 mL d. as $n \to \infty, C \to 0$

Section 3.5

1. Domain $(4,\infty)$ Inverse $f^{-1}(x) = \sqrt{x} + 4$

3. Domain $(-\infty,0)$ Inverse $f^{-1}(x) = -\sqrt{12-x}$

5. Domain $(-\infty,\infty)$ Inverse $f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{3}}$

7. $f^{-1}(x) = \dfrac{(x-9)^2}{4} + 1$

9. $f^{-1}(x) = \left(\dfrac{x-9}{2}\right)^3$

11. $f^{-1}(x) = \dfrac{2-8x}{x}$

13. $f^{-1}(x) = \dfrac{3-7x}{x-1}$

15. $f^{-1}(x) = \dfrac{5x-4}{3+4x}$

17. 65.574 mph

19. 34.073 mph

21. 14.142 feet

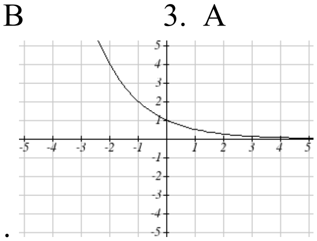
Chapter 4

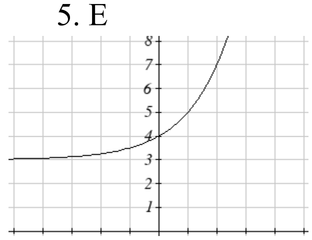
Section 4.1

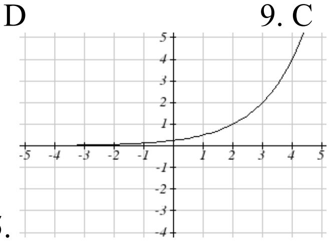
1. Linear
3. Exponential
5. Neither
7. $P(t)=11{,}000(1.085)^{t}$
9. 47622 Fox
11. \$17561.70
13. $y=6(5)^{x}$
15. $y=2000(0.1)^{x}$
17. $y=3(2)^{x}$
19. $y=\left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} = 2.93(0.699)^{x}$
21. $y=\frac{1}{8}(2)^{x}$
23. $34.32\,mg$
25. 1.39%; \$155,368.09
27. \$4,813.55
29. Annual $\approx$ \$7353.84 Quarterly $\approx$ \$7,469.63 Monthly $\approx$ \$7,496.71
    Continuously $\approx$ \$7,510.44
31. 3.03%
33. 7.4 years
35a. $w(t)=(1.113)(1.046)^{t}$ b. \$1.11 c. Below what the model predicts $\approx$ \$5.70

Section 4.2

1. B
3. A
5. E
7. D
9. C

11. 

13. 

15. 

17. $y=4^{x}+4$
19. $y=4^{x+2}$
21. $y=-4^{x}$
23. As $x\to\infty$ $f(x)\to-\infty$. As $x\to-\infty$ $f(x)\to-1$
25. As $x\to\infty$ $f(x)\to-2$ As $x\to-\infty$ $f(x)\to\infty$
27. As $x\to\infty$ $f(x)\to 2$ As $x\to-\infty$ $f(x)\to\infty$
29. $y=-2^{x+2}+1=-4(2)^{x}+1$
31. $y=-2(2)^{-x}+3$
33. $y=-2(3)^{x}+7$
35. $y=2\left(\frac{1}{2}\right)^{x}-4$

Section 4.3

1. $4^{m}=q$
3. $a^{c}=b$
5. $10^{t}=v$
7. $e^{n}=w$
9. $\log_4(y)=x$
11. $\log_c(k)=d$
13. $\log(b)=a$
15. $\ln(h)=k$
17. 9
19. 1/8
21. 1000
23. e^{2}
25. 2
27. -3
29. ½

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31. 4
33. -3
35. -2
37. -1.398
39. 2.708
41. $\frac{\log(14)}{\log(5)} \approx 1.6397$
43. $\frac{\log\left(\frac{1}{15}\right)}{\log(7)} \approx -1.392$
45. $\frac{\ln(17)}{5} \approx 0.567$
47. $\frac{\frac{\log(38)}{\log(3)}+5}{4} \approx 2.078$
49. $\frac{\log(5)}{\log(1.03)} \approx 54.449$
51. $\frac{\log\left(\frac{8}{3}\right)}{3\log(1.04)} \approx 8.335$
53. $\frac{\ln\left(\frac{1}{5}\right)}{-0.12} \approx 13.412$
55. $\frac{\log\left(\frac{5}{8}\right)}{\log\left(\frac{1}{2}\right)} \approx 0.678$
57. $f(t)=300e^{-0.0943t}$
59. $f(t)=10e^{0.03922t}$
61. $f(t)=150(1.0618)^t$
63. $f(t)=50(0.98807)^t$
65. During the year 2013
67. During the year 2074
69. $\approx 34\,hours$
71. 13.532 years

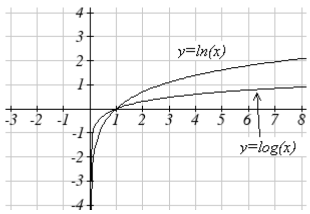
Section 4.4

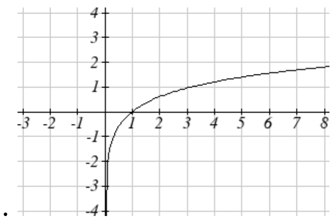
1. $\log_3(4)$
3. $\log_3(7)$
5. $\log_3(5)$
7. $\log_7(2)$
9. $\log(6x^9)$
11. $\ln(2x^7)$
13. $\log\left(x^2(x+1)^3\right)$
15. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$
17. $15\log(x)+13\log(y)-19\log(z)$
19. $-2\ln(a)+4\ln(b)-5\ln(c)$
21. $\frac{3}{2}\log(x)-2\log(y)$
23. $\ln(y)+\frac{1}{2}(\ln(y)-\ln(1-y))$
25. $\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$
27. $x \approx -0.717$
29. $x \approx -6.395$
31. $t \approx 17.329$
33. $x=\frac{2}{7}$
35. $x \approx 0.123$
37. $x \approx 4.642$
39. $x \approx 30.158$
41. $x \approx -2.889$.
43. $x \approx 6.873$ or $x \approx -0.873$
45. $x=\frac{12}{11} \approx 1.091$
47. $x=10$

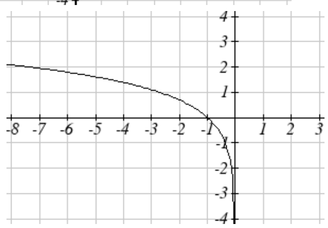
Section 4.5

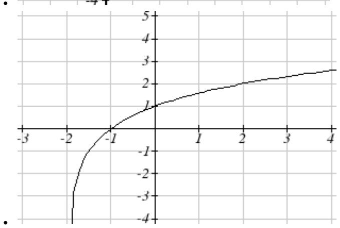
1. Domain: : $x>5$ V. A. @ $x=5$
3. Domain: $x<3$ V.A. @ $x=3$
5. Domain: $x>-\frac{1}{3}$ V.A. @ $x=-\frac{1}{3}$

7. Domain: $x < 0$ V.A. @ $x = 0$

9. 

11. 

13. 

15. 

17. $y = \frac{1}{\log(2)}\log(-(x-1))$

19. $y = -\frac{3}{\log(3)}\log(x+4)$

21. $y = \frac{3}{\log(4)}\log(x+2)$

23. $y = -\frac{2}{\log(5)}\log(-(x-5))$

Section 4.6

1. $f(t) = 13(0.9195)^t$. 2 mg will remain after 22.3098 minutes

3. $f(t) = 200(0.999564)^t$. $f(1000) = 129.3311$ mg

5. r = -0.06448. Initial mass: 9.9018 mg. After 3 days: 0.01648 mg

7. $f(t) = 250(0.9909)^t$. Half-life = 75.8653 minutes

9. $f(t) = a(0.999879)^t$. 60% (0.60a) would remain after 4222.813 years

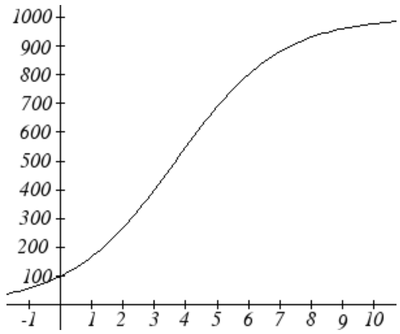
11. $P(t) = 1500(1.02337)^t$ (t in minutes). After 2 hours = 24000. After 100 minutes = 15119

13. a) 610.5143 (about 611) b) 25.6427 minutes c) 10431.21 d) 106.9642 minutes

15. 23.1914 years

17. 53.319 hours

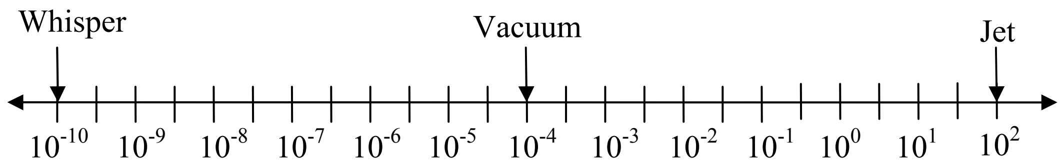
19. $T(t) = 90(0.99166)^t + 75$. a) 134.212 deg b) 112.743 minutes

21. a) 

b) 100 c) 269.487 d) 7.324 years

23. $\log(x) = -0.5$. $x = 0.3162$

25. $\log(x) = 1.5$. $x = 31.623$

27. 

29. 63095.7 times more intense

31. MMS magnitude 5.817

33. a) about 1640671 b) 1.4 hours c) No, because $(2.042727)^{0.693147} \approx e^{0.495105}$

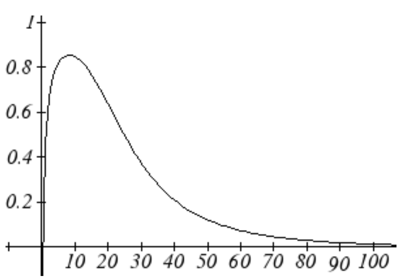
d) Anja's data predicts a continuous growth rate of 0.4116, which is much smaller than the rate 0.495105 you calculated. Our model would overestimate the number of cells.

35. a) The curve that increases rapidly at first is *M(p)*

b) $H(100) = 0.9775$

c) Myoglobin: $M(20) = 0.9524$. Hemoglobin: $H(20) = 0.3242$

d) At 20 torrs: 0.6282. At 40 torrs: 0.2060. At 60 torrs: 0.0714



Efficiency seems to be maximized at about 8 torr

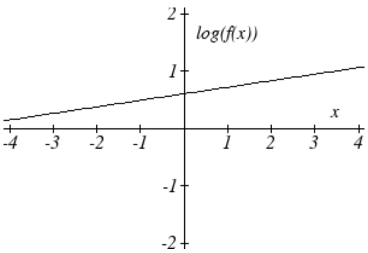
37. a) $C(t) = 1.03526^t$, or $C(t) = e^{0.03466t}$

b) Volume of one cell: $\frac{4}{3}\pi\left(50\times10^{-4}\right)^3 \approx 5.236\times10^{-7}$ cm^3, so will need about 1.9099×10^6 cells for a volume of 1cm^3. $C(t) = 1.9099\times10^6$ after 417.3 hours

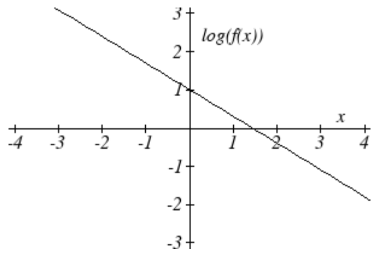
39. 31.699 days

Section 4.7

1. $\log(f(x)) = \log(1.3)x + \log(4)$



3. $\log(f(x)) = \log(0.2)x + 1$



5. $y = e^{\frac{1}{2}x-1} = e^{-1}e^{\frac{1}{2}x} \approx 0.368(1.6487)^x$

7. $y = 10^{-x-2} = 10^{-2}10^{-1x} = 0.01(0.1)^x$

9. $y = 776.682(1.426)^x$

11. $y = 731.92(0.738)^x$

13. Expenditures are approximately \$205

15. $y = 7.599(1.016)^x$ $r = 0.83064$, $y = 0.1493x + 7.4893$, $r = 0.81713$. Using the better function, we predict electricity will be 11.157 cents per kwh

Index

Supplementary Materials

This part of the book contains chapters from the College Algebra text by Stitz and Zeager that will be used for this course. The page numbering and set up of the text will be different.

Supplementary Materials
NOTE: The page numbering is different for this part of the text. You can use the following table of contents to navigate the material.

College Algebra

Version $\lfloor \pi \rfloor = 3$

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July 15, 2011

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While the cover of this textbook lists only two names, the book as it stands today would simply not exist if not for the tireless work and dedication of several people. First and foremost, we wish to thank our families for their patience and support during the creative process. We would also like to thank our students - the sole inspiration for the work. Among our colleagues, we wish to thank Rich Basich, Bill Previts, and Irina Lomonosov, who not only were early adopters of the textbook, but also contributed materials to the project. Special thanks go to Katie Cimperman, Terry Dykstra, Frank LeMay, and Rich Hagen who provided valuable feedback from the classroom. Thanks also to David Stumpf, Ivana Gorgievska, Jorge Gerszonowicz, Kathryn Arocho, Heather Bubnick, and Florin Muscutariu for their unwaivering support (and sometimes defense!) of the project. From outside the classroom, we wish to thank Don Anthan and Ken White, who designed the electric circuit applications used in the text, as well as Drs. Wendy Marley and Marcia Ballinger for the Lorain CCC enrollment data used in the text. The authors are also indebted to the good folks at our schools' bookstores, Gwen Sevtis (Lakeland CC) and Chris Callahan (Lorain CCC), for working with us to get printed copies to the students as inexpensively as possible. We would also like to thank Lakeland folks Jeri Dickinson, Mary Ann Blakeley, Jessica Novak, and Corrie Bergeron for their enthusiasm and promotion of the project. The administration at both schools have also been very supportive of the project, so from Lakeland, we wish to thank Dr. Morris W. Beverage, Jr., President, Dr. Fred Law, Provost, Deans Don Anthan and Dr. Steve Oluic, and the Board of Trustees. From Lorain County Community College, we which to thank Dr. Roy A. Church, Dr. Karen Wells, and the Board of Trustees. From the Ohio Board of Regents, we wish to thank former Chancellor Eric Fingerhut, Darlene McCoy, Associate Vice Chancellor of Affordability and Efficiency, and Kelly Bernard. From OhioLINK, we wish to thank Steve Acker, John Magill, and Stacy Brannan. We also wish to thank the good folks at WebAssign, most notably Chris Hall, COO, and Joel Hollenbeck (former VP of Sales.) Last, but certainly not least, we wish to thank all the folks who have contacted us over the interwebs, most notably Dimitri Moonen and Joel Wordsworth, who gave us great feedback, and Antonio Olivares who helped debug the source code.

Preface

Thank you for your interest in our book, but more importantly, thank you for taking the time to read the Preface. I always read the Prefaces of the textbooks which I use in my classes because I believe it is in the Preface where I begin to understand the authors - who they are, what their motivation for writing the book was, and what they hope the reader will get out of reading the text. Pedagogical issues such as content organization and how professors and students should best use a book can usually be gleaned out of its Table of Contents, but the reasons behind the choices authors make should be shared in the Preface. Also, I feel that the Preface of a textbook should demonstrate the authors' love of their discipline and passion for teaching, so that I come away believing that they really want to help students and not just make money. Thus, I thank my fellow Preface-readers again for giving me the opportunity to share with you the need and vision which guided the creation of this book and passion which both Carl and I hold for Mathematics and the teaching of it.

Carl and I are natives of Northeast Ohio. We met in graduate school at Kent State University in 1997. I finished my Ph.D in Pure Mathematics in August 1998 and started teaching at Lorain County Community College in Elyria, Ohio just two days after graduation. Carl earned his Ph.D in Pure Mathematics in August 2000 and started teaching at Lakeland Community College in Kirtland, Ohio that same month. Our schools are fairly similar in size and mission and each serves a similar population of students. The students range in age from about 16 (Ohio has a Post-Secondary Enrollment Option program which allows high school students to take college courses for free while still in high school.) to over 65. Many of the "non-traditional" students are returning to school in order to change careers. A majority of the students at both schools receive some sort of financial aid, be it scholarships from the schools' foundations, state-funded grants or federal financial aid like student loans, and many of them have lives busied by family and job demands. Some will be taking their Associate degrees and entering (or re-entering) the workforce while others will be continuing on to a four-year college or university. Despite their many differences, our students share one common attribute: they do not want to spend $200 on a College Algebra book.

The challenge of reducing the cost of textbooks is one that many states, including Ohio, are taking quite seriously. Indeed, state-level leaders have started to work with faculty from several of the colleges and universities in Ohio and with the major publishers as well. That process will take considerable time so Carl and I came up with a plan of our own. We decided that the best way to help our students right now was to write our own College Algebra book and give it away electronically for free. We were granted sabbaticals from our respective institutions for the Spring semester of 2009 and actually began writing the textbook on December 16, 2008. Using an open-source text editor called TexNicCenter and an open-source distribution of LaTeX called MikTex 2.7, Carl and I wrote and edited all of the text, exercises and answers and created all of the graphs (using Metapost within LaTeX) for Version $0.\overline{9}$ in about eight months. (We choose to create a text in only black and white to keep printing costs to a minimum for those students who prefer a printed edition. This somewhat Spartan page layout stands in sharp relief to the explosion of colors found in most other College Algebra texts, but neither Carl nor I believe the four-color print adds anything of value.) I used the book in three sections of College Algebra at Lorain County Community College in the Fall of 2009 and Carl's colleague, Dr. Bill Previts, taught a section of College Algebra at Lakeland with the book that semester as well. Students had the option of downloading the book as a .pdf file from our website www.stitz-zeager.com or buying a low-cost printed version from our colleges' respective bookstores. (By giving this book away for free electronically, we end the cycle of new editions appearing every 18 months to curtail the used book market.) During Thanksgiving break in November 2009, many additional exercises written by Dr. Previts were added and the typographical errors found by our students and others were corrected. On December 10, 2009, Version $\sqrt{2}$ was released. The book remains free for download at our website and by using Lulu.com as an on-demand printing service, our bookstores are now able to provide a printed edition for just under $19. Neither Carl nor I have, or will ever, receive any royalties from the printed editions. As a contribution back to the open-source community, all of the LaTeX files used to compile the book are available for free under a Creative Commons License on our website as well. That way, anyone who would like to rearrange or edit the content for their classes can do so as long as it remains free.

The only disadvantage to not working for a publisher is that we don't have a paid editorial staff. What we have instead, beyond ourselves, is friends, colleagues and unknown people in the open-source community who alert us to errors they find as they read the textbook. What we gain in not having to report to a publisher so dramatically outweighs the lack of the paid staff that we have turned down every offer to publish our book. (As of the writing of this Preface, we've had three offers.) By maintaining this book by ourselves, Carl and I retain all creative control and keep the book our own. We control the organization, depth and rigor of the content which means we can resist the pressure to diminish the rigor and homogenize the content so as to appeal to a mass market. A casual glance through the Table of Contents of most of the major publishers' College Algebra books reveals nearly isomorphic content in both order and depth. Our Table of Contents shows a different approach, one that might be labeled "Functions First." To truly use The Rule of Four, that is, in order to discuss each new concept algebraically, graphically, numerically and verbally, it seems completely obvious to us that one would need to introduce functions first. (Take a moment and compare our ordering to the classic "equations first, then the Cartesian Plane and THEN functions" approach seen in most of the major players.) We then introduce a class of functions and discuss the equations, inequalities (with a heavy emphasis on sign diagrams) and applications which involve functions in that class. The material is presented at a level that definitely prepares a student for Calculus while giving them relevant Mathematics which can be used in other classes as well. Graphing calculators are used sparingly and only as a tool to enhance the Mathematics, not to replace it. The answers to nearly all of the computational homework exercises are given in the text and we have gone to great lengths to write some very thought provoking discussion questions whose answers are not given. One will notice that our exercise sets are much shorter than the traditional sets of nearly 100 "drill and kill" questions which build skill devoid of understanding. Our experience has been that students can do about 15-20 homework exercises a night so we very carefully chose smaller sets of questions which cover all of the necessary skills and get the students thinking more deeply about the Mathematics involved.

Critics of the Open Educational Resource movement might quip that "open-source is where bad content goes to die," to which I say this: take a serious look at what we offer our students. Look through a few sections to see if what we've written is bad content in your opinion. I see this open-source book not as something which is "free and worth every penny", but rather, as a high quality alternative to the business as usual of the textbook industry and I hope that you agree. If you have any comments, questions or concerns please feel free to contact me at jeff@stitz-zeager.com or Carl at carl@stitz-zeager.com.

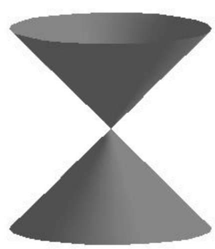
Jeff Zeager
Lorain County Community College
January 25, 2010

Chapter 7

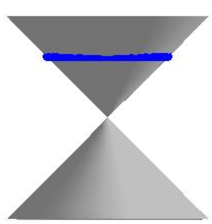
Hooked on Conics

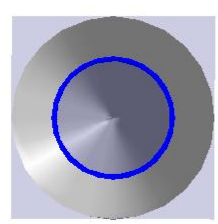
7.1 Introduction to Conics

In this chapter, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

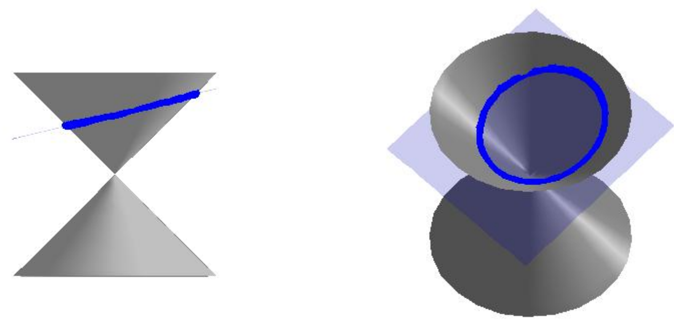


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

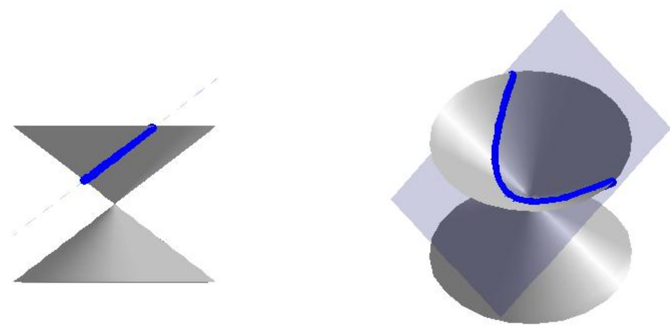




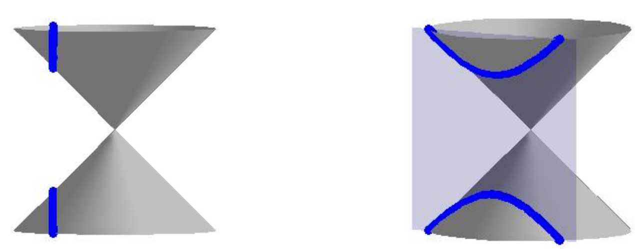
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

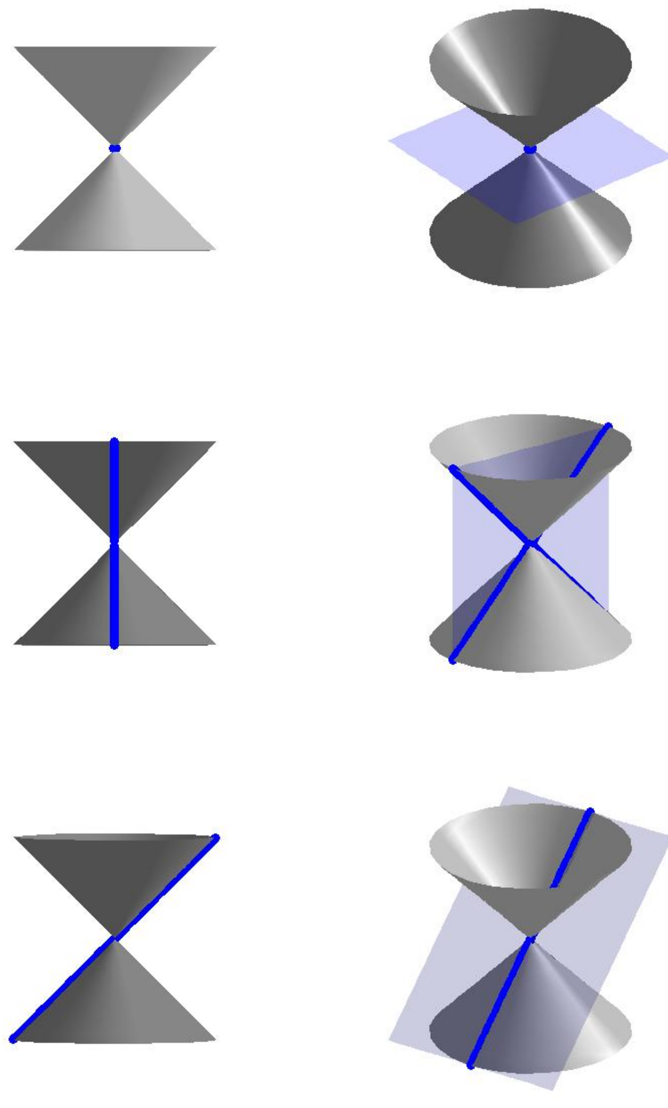


If we slice the cone with a vertical plane, we get a **hyperbola**.



For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

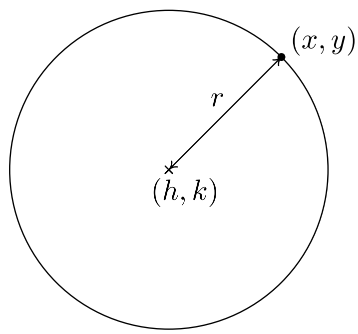


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. To determine equations which describe these curves, we will make use of their definitions in terms of distances.

7.2 CIRCLES

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 7.1. A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Equation 7.1. The Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Example 7.2.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

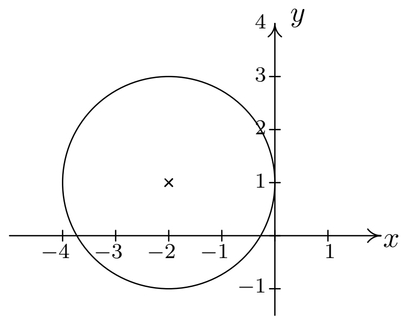
Solution. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

□

Example 7.2.2. Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Solution. From the standard form of a circle, Equation 7.1, we have that $x+2$ is $x-h$, so $h = -2$ and $y-1$ is $y-k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2 + (y-1)^2 = 4$, we'd be contending with $x^2 + 4x + y^2 - 2y + 1 = 0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 7.2.3. Complete the square to find the center and radius of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$.

Solution.

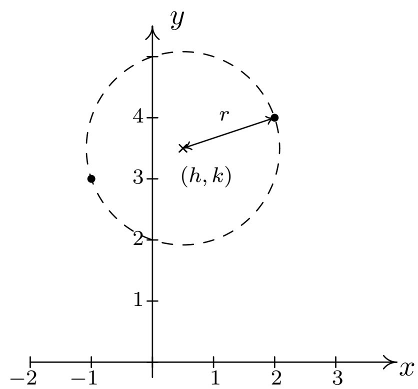
$$\begin{aligned}
3x^2 - 6x + 3y^2 + 4y - 4 &= 0 \\
3x^2 - 6x + 3y^2 + 4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) &= 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x,\ y \\
3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2 + \left(y + \frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}$$

From Equation 7.1, we identify $x - 1$ as $x - h$, so $h = 1$, and $y + \frac{2}{3}$ as $y - k$, so $k = -\frac{2}{3}$. Hence, the center is $(h,k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

Example 7.2.4. Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right)\\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$ □

We close this section with the most important[1] circle in all of mathematics: the **Unit Circle**.

Definition 7.2. The **Unit Circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 7.2.5. Find the points on the unit circle with y-coordinate $\dfrac{\sqrt{3}}{2}$.

Solution. We replace y with $\dfrac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4} + x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. □

[1]While this may seem like an opinion, it is indeed a fact. See Chapters 10 and 11 for details.

7.2.1 Exercises

In Exercises 1 - 6, find the standard equation of the circle and then graph it.

1. Center $(-1,-5)$, radius 10
2. Center $(4,-2)$, radius 3
3. Center $\left(-3,\frac{7}{13}\right)$, radius $\frac{1}{2}$
4. Center $(5,-9)$, radius $\ln(8)$
5. Center $\left(-e,\sqrt{2}\right)$, radius π
6. Center (π,e^2), radius $\sqrt[3]{91}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

7. $x^2-4x+y^2+10y=-25$
8. $-2x^2-36x-2y^2-112=0$
9. $x^2+y^2+8x-10y-1=0$
10. $x^2+y^2+5x-y-1=0$
11. $4x^2+4y^2-24y+36=0$
12. $x^2+x+y^2-\frac{6}{5}y=1$

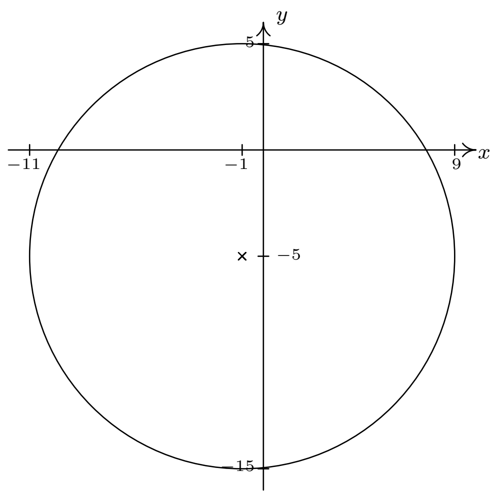
In Exercises 13 - 16, find the standard equation of the circle which satisfies the given criteria.

13. center $(3,5)$, passes through $(-1,-2)$
14. center $(3,6)$, passes through $(-1,4)$
15. endpoints of a diameter: $(3,6)$ and $(-1,4)$
16. endpoints of a diameter: $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$
17. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[2] Find an equation for the wheel assuming that its center lies on the y-axis.
18. Verify that the following points lie on the Unit Circle: $(\pm 1,0)$, $(0,\pm 1)$, $\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2},\pm\frac{1}{2}\right)$
19. Discuss with your classmates how to obtain the standard equation of a circle, Equation 7.1, from the equation of the Unit Circle, $x^2+y^2=1$ using the transformations discussed in Section 1.7. (Thus every circle is just a few transformations away from the Unit Circle.)
20. Find an equation for the function represented graphically by the top half of the Unit Circle. Explain how the transformations is Section 1.7 can be used to produce a function whose graph is either the top or bottom of an arbitrary circle.
21. Find a one-to-one function whose graph is half of a circle. (Hint: Think piecewise.)

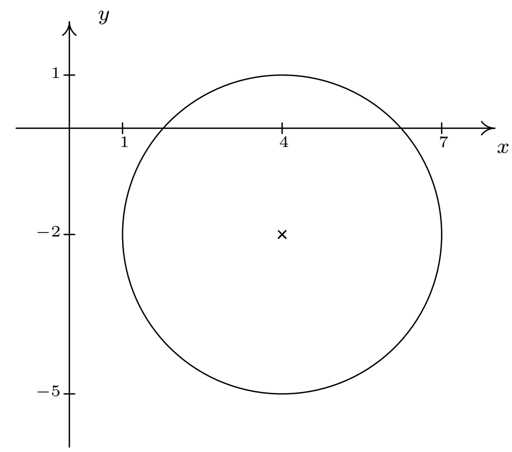
[2]Source: Cedar Point's webpage.

7.2.2 Answers

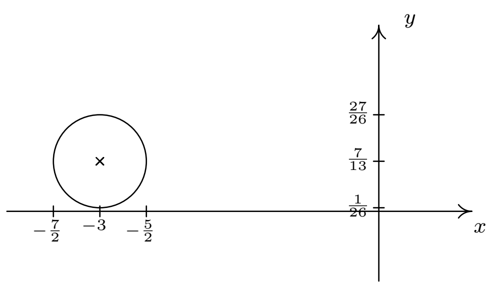
1. $(x+1)^2 + (y+5)^2 = 100$



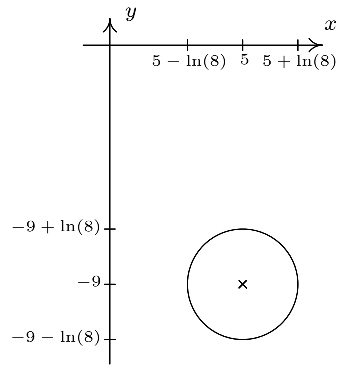
2. $(x-4)^2 + (y+2)^2 = 9$



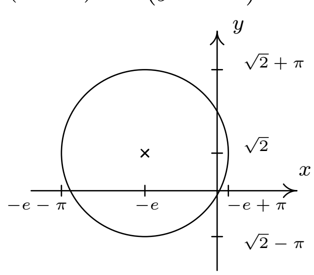
3. $(x+3)^2 + \left(y - \frac{7}{13}\right)^2 = \frac{1}{4}$



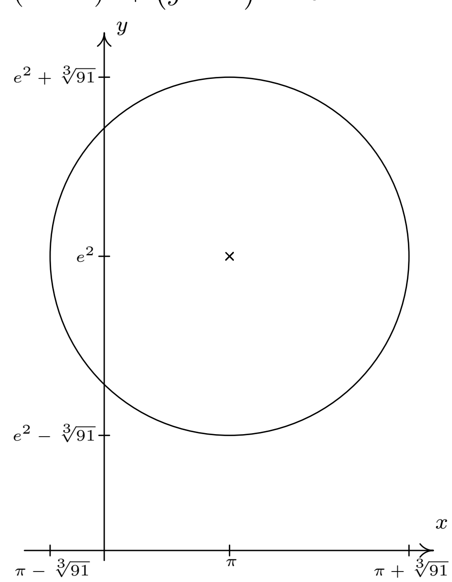
4. $(x-5)^2 + (y+9)^2 = (\ln(8))^2$



5. $(x+e)^2 + \left(y - \sqrt{2}\right)^2 = \pi^2$



6. $(x-\pi)^2 + \left(y - e^2\right)^2 = 91^{\frac{2}{3}}$



7. $(x-2)^2+(y+5)^2=4$
Center $(2,-5)$, radius $r=2$

8. $(x+9)^2+y^2=25$
Center $(-9,0)$, radius $r=5$

9. $(x+4)^2+(y-5)^2=42$
Center $(-4,5)$, radius $r=\sqrt{42}$

10. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

11. $x^2+(y-3)^2=0$
This is not a circle.

12. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

13. $(x-3)^2+(y-5)^2=65$

14. $(x-3)^2+(y-6)^2=20$

15. $(x-1)^2+(y-5)^2=5$

16. $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$

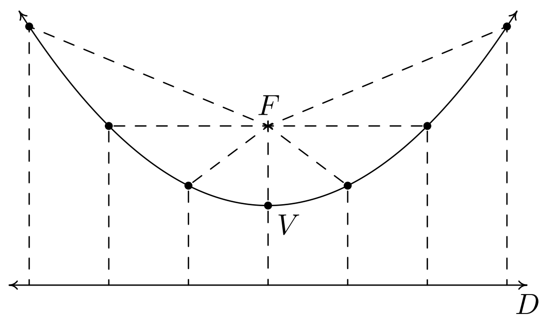
17. $x^2+(y-72)^2=4096$

7.3 Parabolas

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$ ($a \neq 0$) is called a **parabola**. To our surprise and delight, we may also define parabolas in terms of distance.

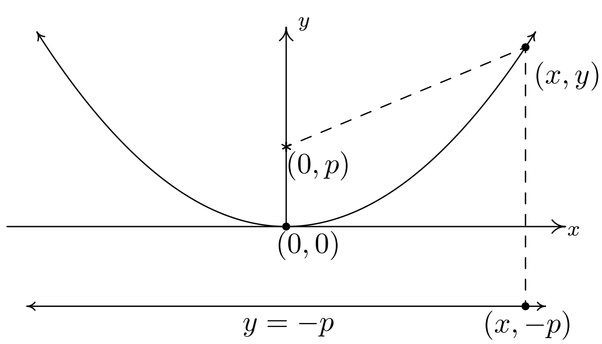
Definition 7.3. Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 2.3. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation 1.1, we get

[1]We'll talk more about what 'directed' means later.

$$\begin{aligned} \sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} & \\ \sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} & \\ x^2+(y-p)^2 &= (y+p)^2 & \text{square both sides} \\ x^2+y^2-2py+p^2 &= y^2+2py+p^2 & \text{expand quantities} \\ x^2 &= 4py & \text{gather like terms} \end{aligned}$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 2.4 with $a = \frac{1}{4p}$ and vertex $(0,0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formula using either transformations from Section 1.7 or re-deriving the formula from Definition 7.3.

Equation 7.2. The Standard Equation of a Vertical[a] Parabola: The equation of a (vertical) parabola with vertex (h,k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

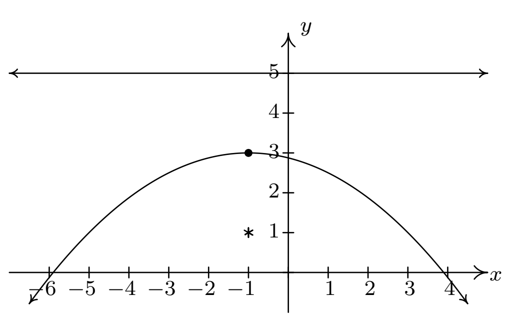
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

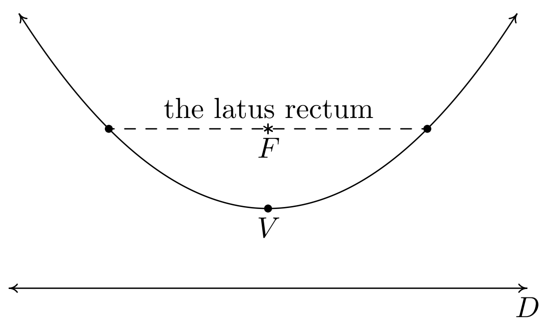
Example 7.3.1. Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.2. Here, $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 3$ so $k = 3$. Hence, the vertex is $(-1,3)$. We also see that $4p = -8$ so $p = -2$. Since $p < 0$, the focus will be below the vertex and the parabola will open downwards.



The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 1)$. The directrix, then, is 2 units above the vertex, so it is the line $y = 5$. □

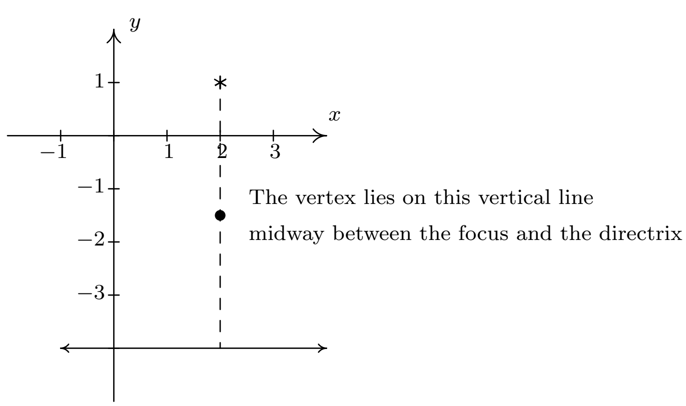
Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[2] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[3] that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p|$, which, in light of Equation 7.2, is easy to find. In our last example, for instance, when graphing $(x + 1)^2 = -8(y - 3)$, we can use the fact that the focal diameter is $|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

Example 7.3.2. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

Solution. Sketching the data yields,



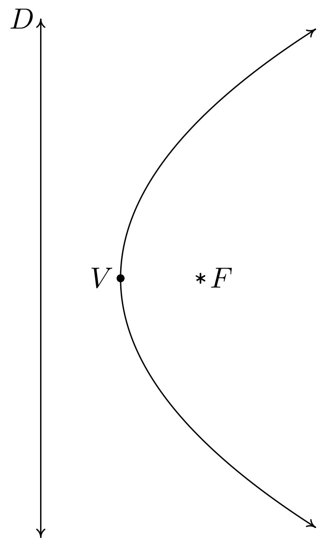
[2] No, I'm not making this up.
[3] Consider this an exercise to show what follows.

From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1 - (-4) = 5$, which means the vertex lies $\frac{5}{2}$ units (halfway) below the focus. Starting at $(2, 1)$ and moving down $5/2$ units leaves us at $(2, -3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p = 5/2$. Plugging all of this data into Equation 7.2 give us

$$\begin{aligned} (x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right) \\ (x-2)^2 &= 10\left(y + \frac{3}{2}\right) \end{aligned}$$

□

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[4] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Equation 7.3. The Standard Equation of a Horizontal Parabola: The equation of a (horizontal) parabola with vertex (h, k) and focal length $|p|$ is

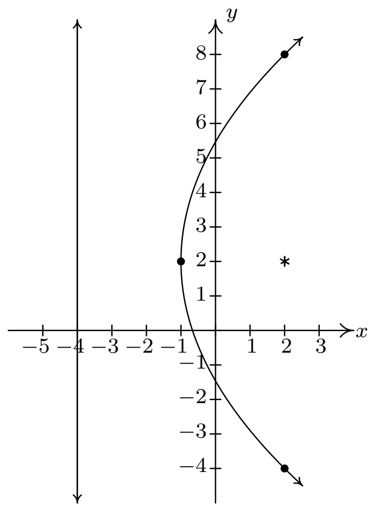
$$(y - k)^2 = 4p(x - h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

[4]plural of 'directrix'

Example 7.3.3. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.3. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p|=3$, which means the focus is 3 units to the right. If we start at $(-1,2)$ and move right 3 units, we arrive at the focus $(2,2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1,2)$, we'd be on the vertical line $x=-4$. Since the focal diameter is $|4p|=12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



□

As with circles, not all parabolas will come to us in the forms in Equations 7.2 or 7.3. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

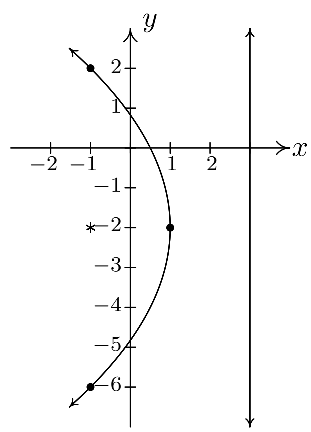
1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 7.3.4. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

Solution. We need a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

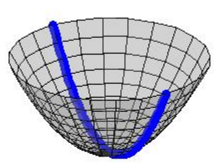
$$\begin{aligned} y^2+4y+8x &= 4 \\ y^2+4y &= -8x+4 \\ y^2+4y+4 &= -8x+4+4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x+8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

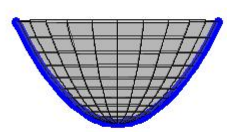
Now that the equation is in the form given in Equation 7.3, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+2$ so $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$. Since $p<0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units to the left of 1, so if we start at $(1,-2)$ and move left 2 units, we arrive at the focus $(-1,-2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1,-2)$, we'd be on the vertical line $x=3$. Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



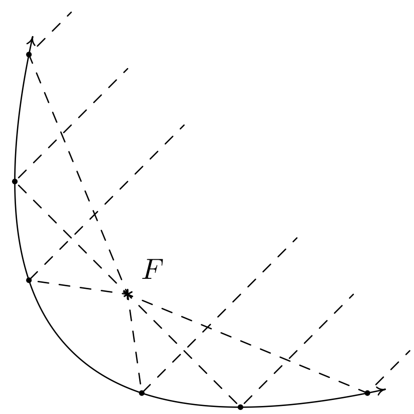
□

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.



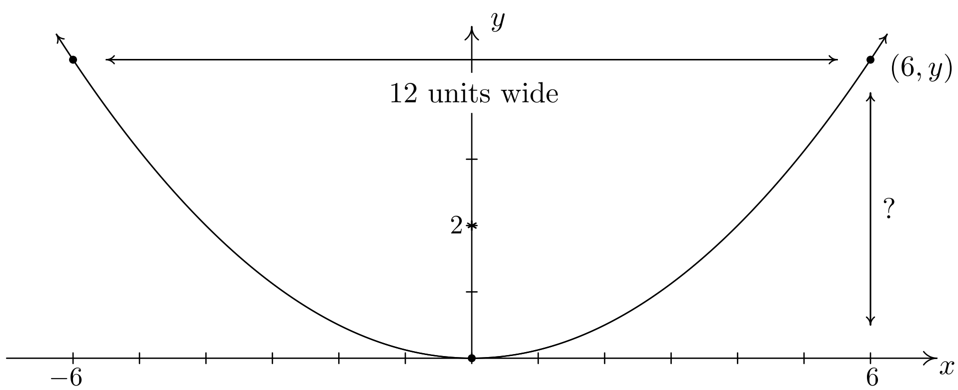


Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



Example 7.3.5. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep. $\square$

7.3.1 Exercises

In Exercises 1 - 8, sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form and identify the vertex, focus and directrix.

9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

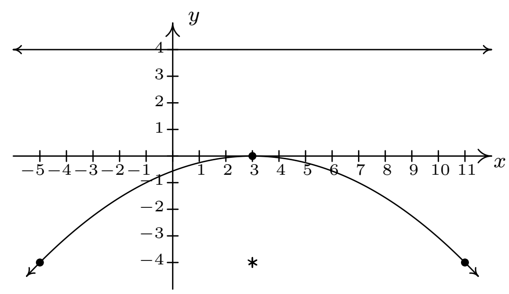
In Exercises 15 - 18, find an equation for the parabola which fits the given criteria.

15. Vertex $(7,0)$, focus $(0,0)$
16. Focus $(10,1)$, directrix $x = 5$
17. Vertex $(-8,-9)$; $(0,0)$ and $(-16,0)$ are points on the curve
18. The endpoints of latus rectum are $(-2,-7)$ and $(4,-7)$
19. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?
20. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape.[5] If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?
21. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.
22. A popular novelty item is the 'mirage bowl.' Follow this link to see another startling application of the reflective property of the parabola.
23. With the help of your classmates, research spinning liquid mirrors. To get you started, check out this website.

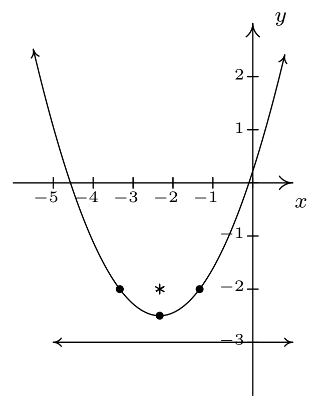
[5] This shape is called a 'parabolic cylinder.'

7.3.2 Answers

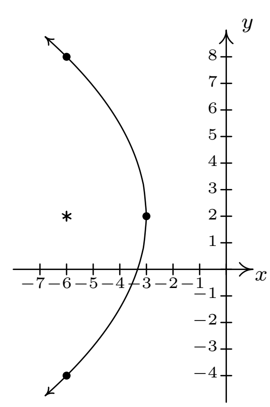
1. $(x-3)^2 = -16y$
   Vertex $(3,0)$
   Focus $(3,-4)$
   Directrix $y=4$
   Endpoints of latus rectum $(-5,-4)$, $(11,-4)$



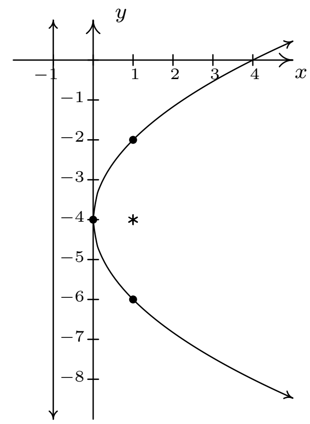
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3},-\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3},-2\right)$
   Directrix $y=-3$
   Endpoints of latus rectum $\left(-\frac{10}{3},-2\right)$, $\left(-\frac{4}{3},-2\right)$



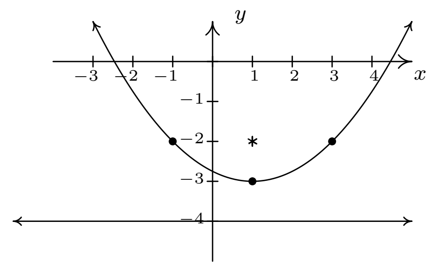
3. $(y-2)^2 = -12(x+3)$
   Vertex $(-3,2)$
   Focus $(-6,2)$
   Directrix $x=0$
   Endpoints of latus rectum $(-6,8)$, $(-6,-4)$



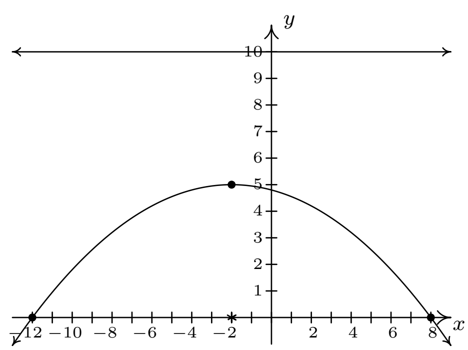
4. $(y+4)^2 = 4x$
   Vertex $(0, -4)$
   Focus $(1, -4)$
   Directrix $x = -1$
   Endpoints of latus rectum $(1, -2)$, $(1, -6)$



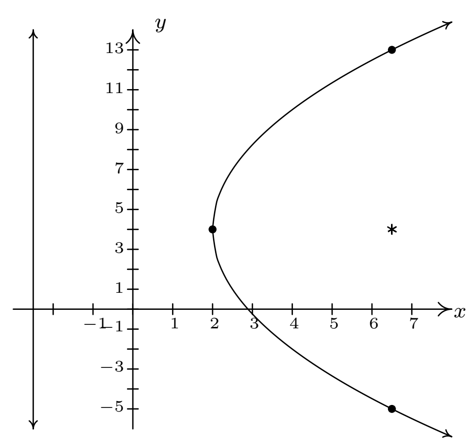
5. $(x-1)^2 = 4(y+3)$
   Vertex $(1, -3)$
   Focus $(1, -2)$
   Directrix $y = -4$
   Endpoints of latus rectum $(3, -2)$, $(-1, -2)$



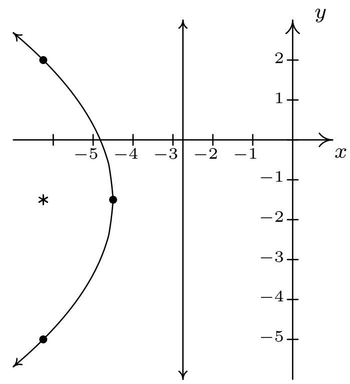
6. $(x+2)^2 = -20(y-5)$
   Vertex $(-2, 5)$
   Focus $(-2, 0)$
   Directrix $y = 10$
   Endpoints of latus rectum $(-12, 0)$, $(8, 0)$



7. $(y-4)^2 = 18(x-2)$
   Vertex $(2, 4)$
   Focus $\left(\frac{13}{2}, 4\right)$
   Directrix $x = -\frac{5}{2}$
   Endpoints of latus rectum $\left(\frac{13}{2}, -5\right)$, $\left(\frac{13}{2}, 13\right)$



8. $\left(y+\frac{3}{2}\right)^2=-7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2},-\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4},-\frac{3}{2}\right)$
Directrix $x=-\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4},2\right)$, $\left(-\frac{25}{4},-5\right)$



9. $(y-5)^2=27(x-4)$
Vertex $(4,5)$
Focus $\left(\frac{43}{4},5\right)$
Directrix $x=-\frac{11}{4}$

10. $\left(x+\frac{2}{5}\right)^2=-\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5},1\right)$
Focus $\left(-\frac{2}{5},\frac{19}{20}\right)$
Directrix $y=\frac{21}{20}$

11. $(x+1)^2=8(y-6)$
Vertex $(-1,6)$
Focus $(-1,8)$
Directrix $y=4$

12. $(y+1)^2=-\frac{1}{2}(x-10)$
Vertex $(10,-1)$
Focus $\left(\frac{79}{8},-1\right)$
Directrix $x=\frac{81}{8}$

13. $(x-5)^2=-12(y-2)$
Vertex $(5,2)$
Focus $(5,-1)$
Directrix $y=5$

14. $\left(y-\frac{9}{2}\right)^2=-\frac{4}{3}(x-2)$
Vertex $\left(2,\frac{9}{2}\right)$
Focus $\left(\frac{5}{3},\frac{9}{2}\right)$
Directrix $x=\frac{7}{3}$

15. $y^2=-28(x-7)$

16. $(y-1)^2=10\left(x-\frac{15}{2}\right)$

17. $(x+8)^2=\frac{64}{9}(y+9)$

18. $(x-1)^2=6\left(y+\frac{17}{2}\right)$ or
$(x-1)^2=-6\left(y+\frac{11}{2}\right)$

19. The bulb should be placed 0.625 centimeters above the vertex of the mirror. (As verified by Carl himself!)

20. The receiver should be placed 5.0625 centimeters from the vertex of the cross section of the antenna.

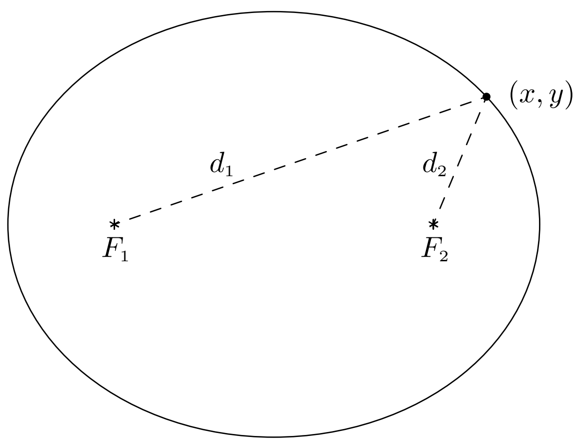
21. The arch can be modeled by $x^2=-(y-9)$ or $y=9-x^2$. One foot in from the base of the arch corresponds to either $x=\pm2$, so the height is $y=9-(\pm2)^2=5$ feet.

7.4 Ellipses

In the definition of a circle, Definition 7.1, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 7.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

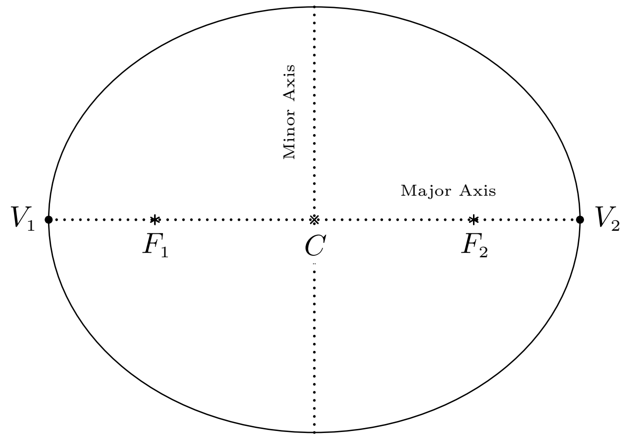
[a] the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

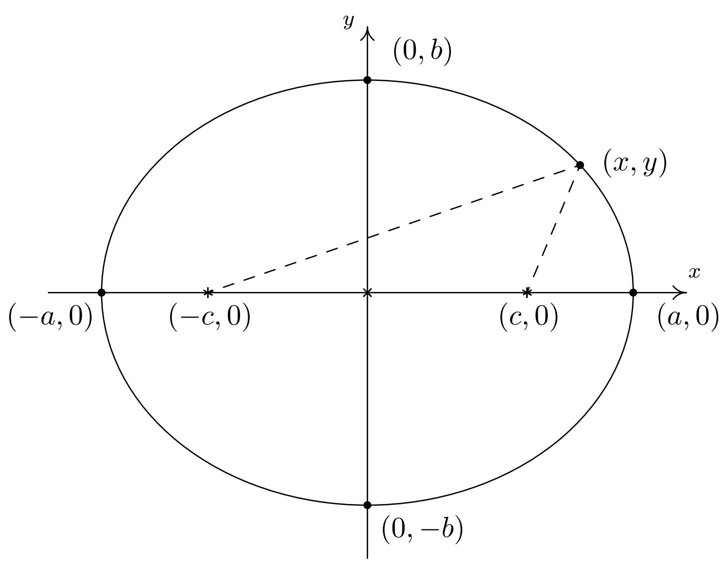
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 7.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)+(a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0, b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\
\sqrt{(0-(-c))^2+(b-0)^2} + \sqrt{(0-c)^2+(b-0)^2} &= 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\
2\sqrt{b^2+c^2} &= 2a \\
\sqrt{b^2+c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x, y) on the ellipse. Applying Definition 7.4, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2 - cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h, k), we could use the transformations from Section 1.7 or re-derive the equation using Definition 7.4 and the distance formula to obtain the formula below.

Equation 7.4. The Standard Equation of an Ellipse: For positive unequal numbers a and b, the equation of an ellipse with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some remarks about Equation 7.4 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle, Equation 7.1, and divide both sides by r^2, we get

Equation 7.5. The Alternate Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is

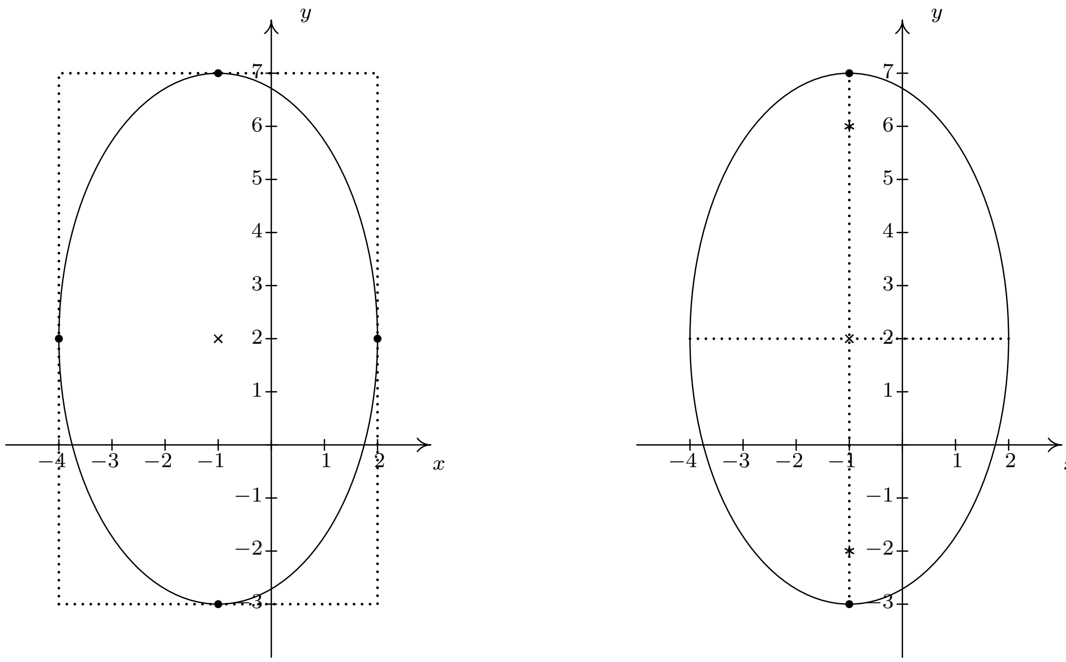
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Notice the similarity between Equation 7.4 and Equation 7.5. Both equations involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both Equations 7.4 and 7.5 as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition 7.2. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.[1]

Example 7.4.1. Graph $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 1$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution. We see that this equation is in the standard form of Equation 7.4. Here $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 2$ so $k = 2$. Hence, our ellipse is centered at $(-1, 2)$. We see that $a^2 = 9$ so $a = 3$, and $b^2 = 25$ so $b = 5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.

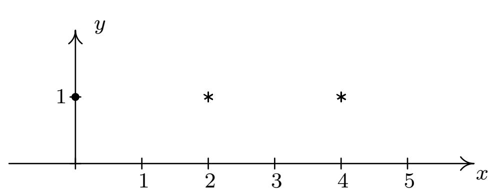
[1] This was foreshadowed in Exercise 19 in Section 7.2.



Since we moved farther in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, which means the minor axis lies along the horizontal line, $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.) To find the foci, we find $c = \sqrt{25 - 9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, -2)$ and $(-1, 6)$. Plotting all this information gives the graph seen above on the right. □

Example 7.4.2. Find the equation of the ellipse with foci $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

Solution. Plotting the data given to us, we have



From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, we know it is $(3, 1)$. Since one vertex is $(0, 1)$ we have that $a = 3$, so $a^2 = 9$. All that remains is to find b^2. Since the foci are 1 unit away from the center, we know $c = 1$. Since $a > b$, we have $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$, so $b^2 = 8$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get our final answer to be $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1$. □

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of Equation 7.4. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1.

Example 7.4.3. Graph $x^2 + 4y^2 - 2x + 24y + 33 = 0$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

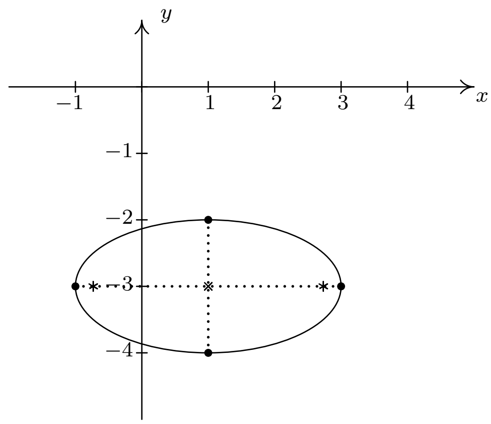
Solution. Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[2] We need to complete both squares, and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned} x^2 + 4y^2 - 2x + 24y + 33 &= 0 \\ x^2 - 2x + 4y^2 + 24y &= -33 \\ x^2 - 2x + 4\left(y^2 + 6y\right) &= -33 \\ \left(x^2 - 2x + 1\right) + 4\left(y^2 + 6y + 9\right) &= -33 + 1 + 4(9) \\ (x-1)^2 + 4(y+3)^2 &= 4 \\ \frac{(x-1)^2 + 4(y+3)^2}{4} &= \frac{4}{4} \\ \frac{(x-1)^2}{4} + (y+3)^2 &= 1 \\ \frac{(x-1)^2}{4} + \frac{(y+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.4, we see that $x - h$ is $x - 1$ so $h = 1$, and $y - k$ is $y + 3$ so $k = -3$. Hence, our ellipse is centered at $(1, -3)$. We see that $a^2 = 4$ so $a = 2$, and $b^2 = 1$ so $b = 1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the x direction than in the y direction, the major axis will lie along the horizontal line $y = -3$, which means the minor axis lies along the vertical line $x = 1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, -3)$ and $(3, -3)$, and the endpoints of the minor axis are $(1, -2)$ and $(1, -4)$. To find the foci, we find $c = \sqrt{4-1} = \sqrt{3}$, which means

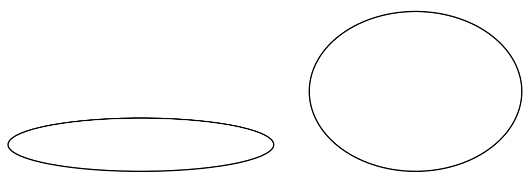
[2]The equation of a parabola has only one squared variable and the equation of a circle has two squared variables with *identical* coefficients.

the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1 - \sqrt{3}, -3)$ and $(1 + \sqrt{3}, -3)$. Plotting all of this information gives



□

As you come across ellipses in the homework exercises and in the wild, you'll notice they come in all shapes in sizes. Compare the two ellipses below.



Certainly, one ellipse is more round than the other. This notion of 'roundness' is quantified below.

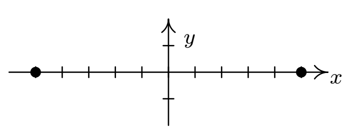
Definition 7.5. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.98$; for the ellipse above on the right, $e \approx 0.66$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse; the closer the eccentricity is to 1, the more 'eccentric' the ellipse.

Example 7.4.4. Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution. As before, we plot the data given to us

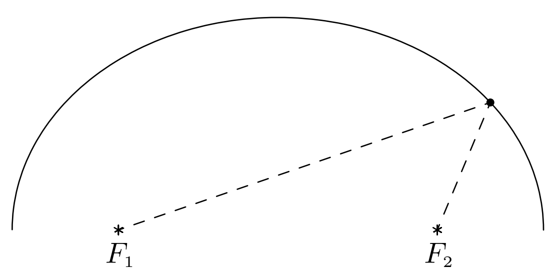


From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(\pm 5, 0)$, we get $a = 5$ so $a^2 = 25$. We also know that the center is $(0, 0)$ because the center is the midpoint of the vertices. All that remains is to find b^2. To that end, we use the fact that the eccentricity $e = \frac{1}{4}$ which means

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a} = \frac{c}{5} = \frac{1}{4}$$

from which we get $c = \frac{5}{4}$. To get b^2, we use the fact that $c = \sqrt{a^2 - b^2}$, so $\frac{5}{4} = \sqrt{25 - b^2}$ from which we get $b^2 = \frac{375}{16}$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$. □

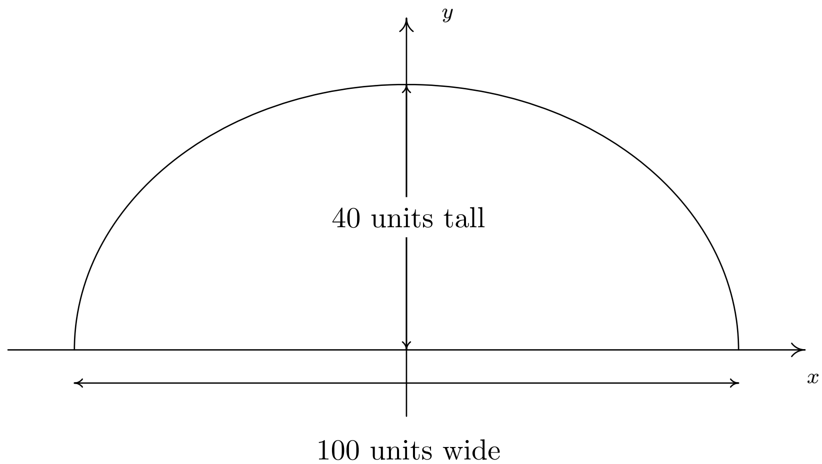
As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 7.4.5. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields



It's most convenient to imagine this ellipse centered at $(0,0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation $\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1$. We're looking for the foci, and we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, so that the foci are 30 units from the center. That means they are $50 - 30 = 20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery. $\square$

7.4.1 Exercises

In Exercises 1 - 8, graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

9. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
10. $12x^2 + 3y^2 - 30y + 39 = 0$
11. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
12. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
13. $9x^2 + 4y^2 - 4y - 8 = 0$
14. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$

In Exercises 15 - 20, find the standard form of the equation of the ellipse which has the given properties.

15. Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
16. Foci $(0, \pm 5)$, Vertices $(0, \pm 8)$.
17. Foci $(\pm 3, 0)$, length of the Minor Axis 10
18. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Minor Axis $(8, 4)$, $(8, 0)$
19. Center $(5, 2)$, Vertex $(0, 2)$, eccentricity $\frac{1}{2}$
20. All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$. (One might also say that the ellipse is "tangent to the axes" at those two points.)
21. Repeat Example 7.4.5 for a whispering gallery 200 feet wide and 75 feet tall.
22. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 21 in Section 7.3.

23. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

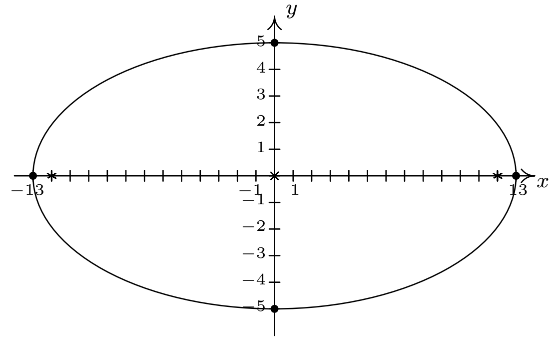
24. The graph of an ellipse clearly fails the Vertical Line Test, Theorem 1.1, so the equation of an ellipse does not define y as a function of x. However, much like with circles and horizontal parabolas, we can split an ellipse into a top half and a bottom half, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the ellipses given in Exercises 1 - 8 above. What difficulties arise when you plot them on the calculator?

25. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, National Statuary Hall in Washington, D.C., and The Cincinnati Museum Center. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in Example 7.4.5?

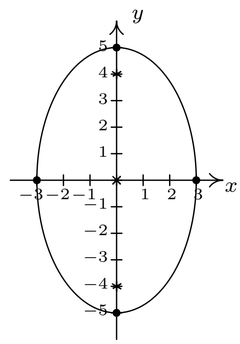
26. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

7.4.2 Answers

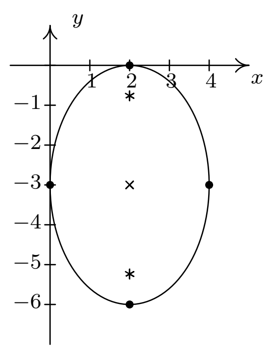
1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $y = 0$
   Minor axis along $x = 0$
   Vertices $(13,0)$, $(-13,0)$
   Endpoints of Minor Axis $(0,-5)$, $(0,5)$
   Foci $(12,0)$, $(-12,0)$
   $e = \frac{12}{13}$



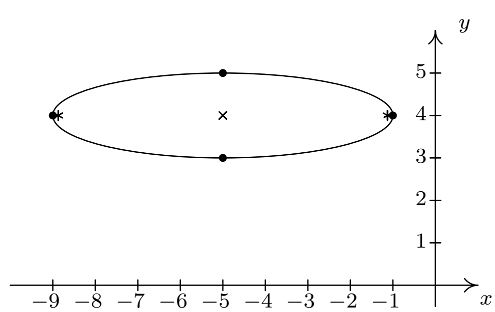
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $x = 0$
   Minor axis along $y = 0$
   Vertices $(0,5)$, $(0,-5)$
   Endpoints of Minor Axis $(-3,0)$, $(3,0)$
   Foci $(0,-4)$, $(0,4)$
   $e = \frac{4}{5}$



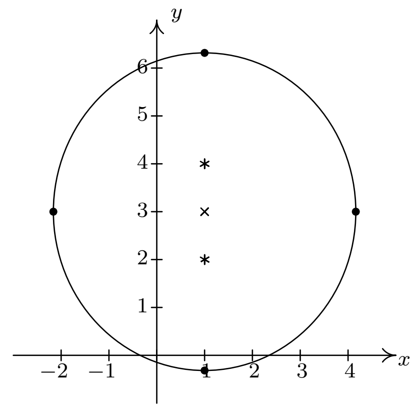
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Major axis along $x = 2$
   Minor axis along $y = -3$
   Vertices $(2,0)$, $(2,-6)$
   Endpoints of Minor Axis $(0,-3)$, $(4,-3)$
   Foci $(2,-3+\sqrt{5})$, $(2,-3-\sqrt{5})$
   $e = \frac{\sqrt{5}}{3}$



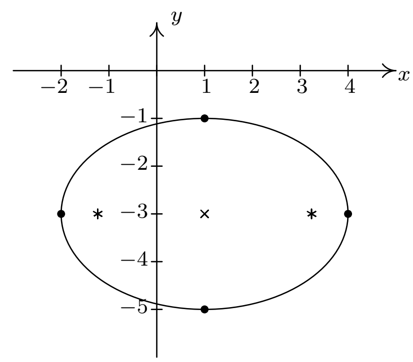
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
   Center $(-5,4)$
   Major axis along $y = 4$
   Minor axis along $x = -5$
   Vertices $(-9,4)$, $(-1,4)$
   Endpoints of Minor Axis $(-5,3)$, $(-5,5)$
   Foci $(-5+\sqrt{15},4)$, $(-5-\sqrt{15},4)$
   $e = \frac{\sqrt{15}}{4}$



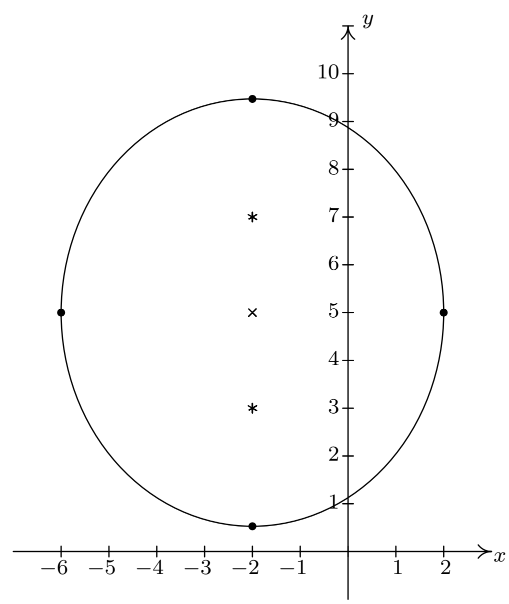
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1, 3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Endpoints of the Minor Axis
$(1-\sqrt{10}, 3)$, $(1+\sqrt{10}, 3)$
Foci $(1, 2)$, $(1, 4)$
$e = \frac{\sqrt{11}}{11}$



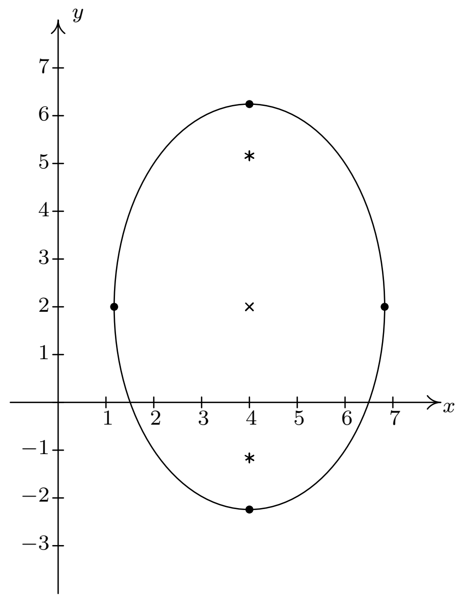
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1, -3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4, -3)$, $(-2, -3)$
Endpoints of the Minor Axis $(1, -1)$, $(1, -5)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \frac{\sqrt{5}}{3}$



7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
Center $(-2, 5)$
Major axis along $x = -2$
Minor axis along $y = 5$
Vertices $(-2, 5+2\sqrt{5})$, $(-2, 5-2\sqrt{5})$
Endpoints of the Minor Axis $(-6, 5)$, $(2, 5)$
Foci $(-2, 7)$, $(-2, 3)$
$e = \frac{\sqrt{5}}{5}$



8. $\dfrac{(x-4)^2}{8}+\dfrac{(y-2)^2}{18}=1$
Center $(4,2)$
Major axis along $x=4$
Minor axis along $y=2$
Vertices $(4,2+3\sqrt{2})$, $(4,2-3\sqrt{2})$
Endpoints of the Minor Axis
$(4-2\sqrt{2},2)$, $(4+2\sqrt{2},2)$
Foci $(4,2+\sqrt{10})$, $(4,2-\sqrt{10})$
$e=\frac{\sqrt{5}}{3}$



9. $\dfrac{(x-3)^2}{25}+\dfrac{(y-1)^2}{9}=1$
Center $(3,1)$
Major Axis along $y=1$
Minor Axis along $x=3$
Vertices $(8,1)$, $(-2,1)$
Endpoints of Minor Axis $(3,4)$, $(3,-2)$
Foci $(7,1)$, $(-1,1)$
$e=\frac{4}{5}$

10. $\dfrac{x^2}{3}+\dfrac{(y-5)^2}{12}=1$
Center $(0,5)$
Major axis along $x=0$
Minor axis along $y=5$
Vertices $(0,5-2\sqrt{3}),(0,5+2\sqrt{3})$
Endpoints of Minor Axis $(-\sqrt{3},5)$, $(\sqrt{3},5)$
Foci $(0,2),(0,8)$
$e=\frac{\sqrt{3}}{2}$

11. $\dfrac{(x-3)^2}{18}+\dfrac{(y+2)^2}{5}=1$
Center $(3,-2)$
Major axis along $y=-2$
Minor axis along $x=3$
Vertices $(3-3\sqrt{2},-2),(3+3\sqrt{2},-2)$
Endpoints of Minor Axis $(3,-2+\sqrt{5})$, $(3,-2-\sqrt{5})$
Foci $(3-\sqrt{13},-2),(3+\sqrt{13},-2)$
$e=\frac{\sqrt{26}}{6}$

12. $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{8}=1$
Center $(1,3)$
Major Axis along $y=3$
Minor Axis along $x=1$
Vertices $(5,3)$, $(-3,3)$
Endpoints of Minor Axis $(1,3+2\sqrt{2})$, $(1,3-2\sqrt{2})$
Foci $(1+2\sqrt{2},3)$, $(1-2\sqrt{2},3)$
$e=\frac{\sqrt{2}}{2}$

13. $\dfrac{x^2}{1} + \dfrac{4\left(y - \frac{1}{2}\right)^2}{9} = 1$
Center $\left(0, \frac{1}{2}\right)$
Major Axis along $x = 0$ (the y-axis)
Minor Axis along $y = \frac{1}{2}$
Vertices $(0, 2)$, $(0, -1)$
Endpoints of Minor Axis $\left(-1, \frac{1}{2}\right)$, $\left(1, \frac{1}{2}\right)$
Foci $\left(0, \frac{1+\sqrt{5}}{2}\right)$, $\left(0, \frac{1-\sqrt{5}}{2}\right)$
$e = \frac{\sqrt{5}}{3}$

14. $\dfrac{(x-2)^2}{5} + \dfrac{(y+2)^2}{6} = 1$
Center $(2, -2)$
Major Axis along $x = 2$
Minor Axis along $y = -2$
Vertices $\left(2, -2 + \sqrt{6}\right)$, $\left(2, -2 - \sqrt{6}\right)$
Endpoints of Minor Axis $\left(2 - \sqrt{5}, -2\right)$, $\left(2 + \sqrt{5}, -2\right)$
Foci $(2, -1)$, $(2, -3)$
$e = \frac{\sqrt{6}}{6}$

15. $\dfrac{(x-3)^2}{9} + \dfrac{(y-7)^2}{25} = 1$

16. $\dfrac{x^2}{39} + \dfrac{y^2}{64} = 1$

17. $\dfrac{x^2}{34} + \dfrac{y^2}{25} = 1$

18. $\dfrac{(x-8)^2}{25} + \dfrac{(y-2)^2}{4} = 1$

19. $\dfrac{(x-5)^2}{25} + \dfrac{4(y-2)^2}{75} = 1$

20. $\dfrac{(x-8)^2}{64} + \dfrac{(y+9)^2}{81} = 1$

21. Jamie and Jason should stand $100 - 25\sqrt{7} \approx 33.86$ feet from opposite ends of the gallery.

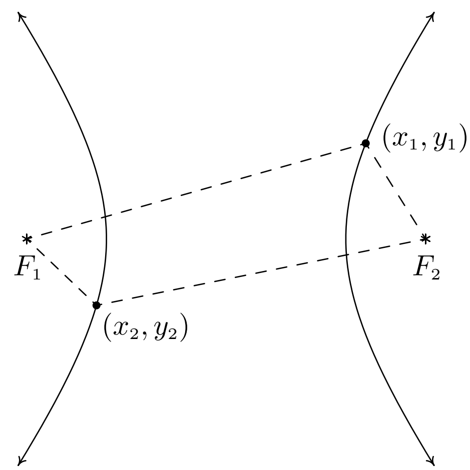
22. The arch can be modeled by the top half of $\frac{x^2}{9} + \frac{y^2}{81} = 1$. One foot in from the base of the arch corresponds to either $x = \pm 2$. Plugging in $x = \pm 2$ gives $y = \pm 3\sqrt{5}$ and since y represents a height, we choose $y = 3\sqrt{5} \approx 6.71$ feet.

23. Distance from the sun to aphelion ≈ 1.0167 AU.
Distance from the sun to perihelion ≈ 0.9833 AU.

7.5 Hyperbolas

In the definition of an ellipse, Definition 7.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 7.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



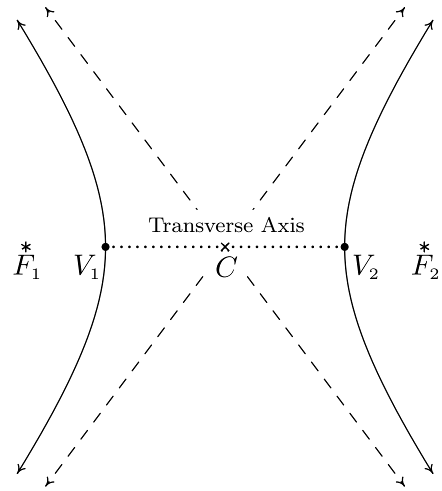
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) = d$$

and

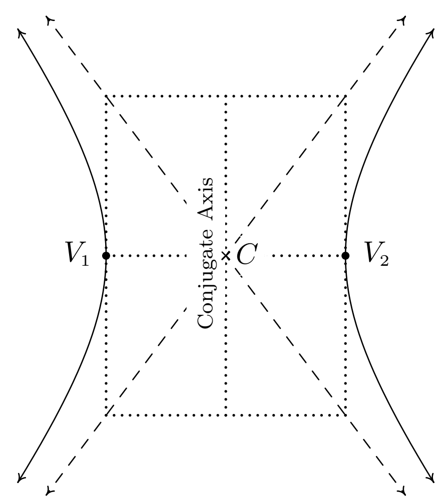
$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) = d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



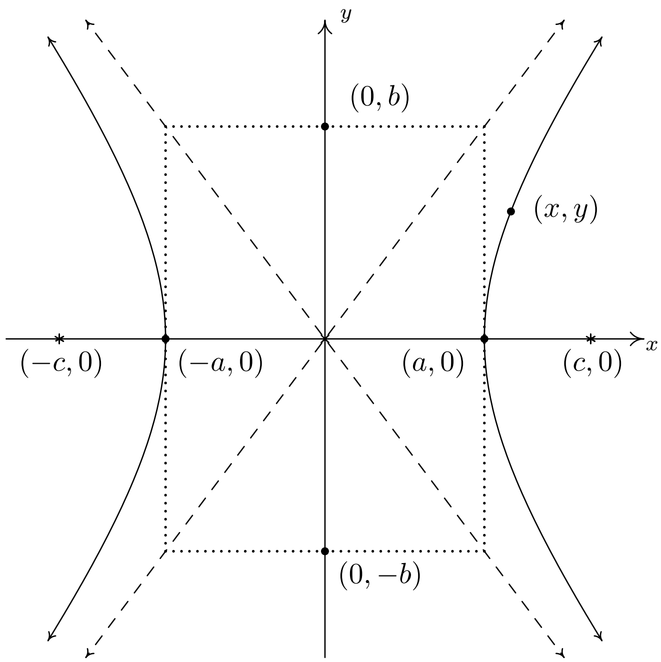
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. In pictures we have



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 7.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0, 0)$, the vertices are $(a, 0)$ and $(-a, 0)$ and the foci are $(c, 0)$ and $(-c, 0)$. We label the endpoints of the conjugate axis $(0, b)$ and $(0, -b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically we have



Since $(a, 0)$ is on the hyperbola, it must satisfy the conditions of Definition 7.6. That is, the distance from $(-c, 0)$ to $(a, 0)$ minus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c) - (c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x, y) on the hyperbola. Applying Definition 7.6, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} &= 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a \end{aligned}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 7.4, we arrive at the following.[1]

[1]It is a good exercise to actually work this out.

$$\left(a^2-c^2\right)x^2+a^2y^2 = a^2\left(a^2-c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a<c$, we get $a^2<c^2$ so that a^2-c^2 is a negative number. Hence, c^2-a^2 is a positive number. For reasons which will become clear soon, we re-write the equation by solving for y^2/x^2 to get

$$\begin{aligned}
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right)\\
-\left(c^2-a^2\right)x^2+a^2y^2 &= -a^2\left(c^2-a^2\right)\\
a^2y^2 &= \left(c^2-a^2\right)x^2-a^2\left(c^2-a^2\right)\\
\frac{y^2}{x^2} &= \frac{\left(c^2-a^2\right)}{a^2}-\frac{\left(c^2-a^2\right)}{x^2}
\end{aligned}$$

As x and y attain very large values, the quantity $\frac{(c^2-a^2)}{x^2}\to 0$ so that $\frac{y^2}{x^2}\to\frac{(c^2-a^2)}{a^2}$. By setting $b^2=c^2-a^2$ we get $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$. This shows that $y\to\pm\frac{b}{a}x$ as $|x|$ grows large. Thus $y=\pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2-c^2=-b^2$ which yields

$$\begin{aligned}
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right)\\
-b^2x^2+a^2y^2 &= -a^2b^2\\
\frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1
\end{aligned}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h,k), we would get the following.

Equation 7.6. The Standard Equation of a Horizontal[a] Hyperbola For positive numbers a and b, the equation of a horizontal hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

[a]That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

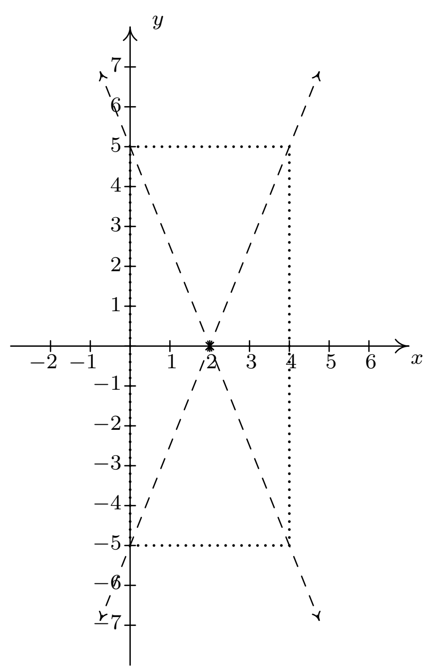
Equation 7.7. The Standard Equation of a Vertical Hyperbola For positive numbers a and b, the equation of a vertical hyperbola with center (h,k) is:

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

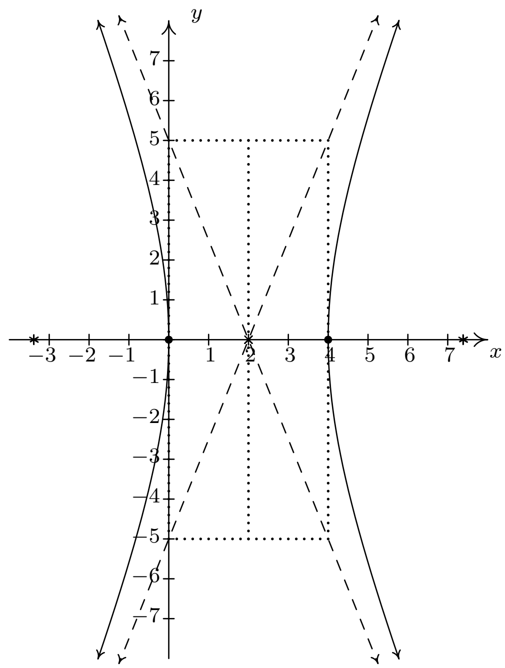
The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, as seen in the derivation, can be found by the formula $c = \sqrt{a^2+b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 7.5.1. Graph the equation $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

Solution. We first see that this equation is given to us in the standard form of Equation 7.6. Here $x-h$ is $x-2$ so $h=2$, and $y-k$ is y so $k=0$. Hence, our hyperbola is centered at $(2,0)$. We see that $a^2=4$ so $a=2$, and $b^2=25$ so $b=5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.



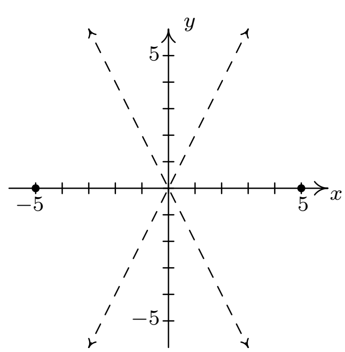
Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the x-axis. Hence, the conjugate axis lies along the vertical line $x=2$. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2,0)$ at $(0,0)$ and $(4,0)$. To find the foci, we need $c = \sqrt{a^2+b^2} = \sqrt{4+25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2,0)$ to arrive at $(2-\sqrt{29},0)$ (approximately $(-3.39,0)$) and $(2+\sqrt{29},0)$ (approximately $(7.39,0)$). To determine the equations of the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(2,0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Using the point-slope equation of a line, Equation 2.2, yields $y - 0 = \pm\frac{5}{2}(x-2)$, so we get $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$. Putting it all together, we get



□

Example 7.5.2. Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution. Plotting the data given to us, we have



This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the vertices are $(\pm 5, 0)$, we have $a = 5$ so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$. □

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side
2. Complete the square in both variables as needed
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1

Example 7.5.3. Consider the equation $9y^2 - x^2 - 6x = 10$. Put this equation in to standard form and graph. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci, and the equations of the asymptotes.

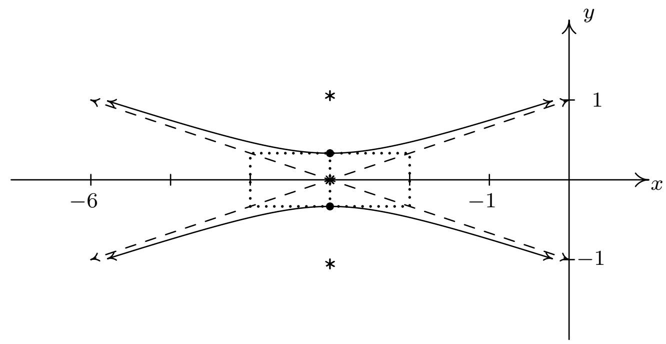
Solution. We need only complete the square on x:

$$\begin{aligned} 9y^2 - x^2 - 6x &= 10 \\ 9y^2 - 1\left(x^2 + 6x\right) &= 10 \\ 9y^2 - \left(x^2 + 6x + 9\right) &= 10 - 1(9) \\ 9y^2 - (x+3)^2 &= 1 \\ \frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.7, we see that $x-h$ is $x+3$ so $h=-3$, and $y-k$ is y so $k=0$. Hence, our hyperbola is centered at $(-3,0)$. We find that $a^2=1$ so $a=1$, and $b^2=\frac{1}{9}$ so $b=\frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle. Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x=-3$ and the conjugate axis lies along the x-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3,0)$ at $\left(-3,\frac{1}{3}\right)$ and $\left(-3,-\frac{1}{3}\right)$. To find the foci, we use

$$c = \sqrt{a^2+b^2} = \sqrt{\frac{1}{9}+1} = \frac{\sqrt{10}}{3}$$

Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3,0)$ to arrive at $\left(-3,\frac{\sqrt{10}}{3}\right)$ and $\left(-3,-\frac{\sqrt{10}}{3}\right)$. To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3,0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Using the point-slope equation of a line, Equation 2.2, we get $y=\frac{1}{3}x+1$ and $y=-\frac{1}{3}x-1$. Putting it all together, we get



□

Hyperbolas can be used in so-called 'trilateration,' or 'positioning' problems. The procedure outlined in the next example is the basis of the (now virtually defunct) LOng Range Aid to Navigation (LORAN for short) system.[2]

Example 7.5.4. Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. If the speed of sound that night is 760 miles per hour, determine a hyperbolic path along which Sasquatch must be located.

Solution. Since Jeff hears Sasquatch sooner, it is closer to Jeff than it is to Carl. Since the speed of sound is 760 miles per hour, we can determine how much closer Sasquatch is to Jeff by multiplying

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 9\,\text{seconds} = 1.9\,\text{miles}$$

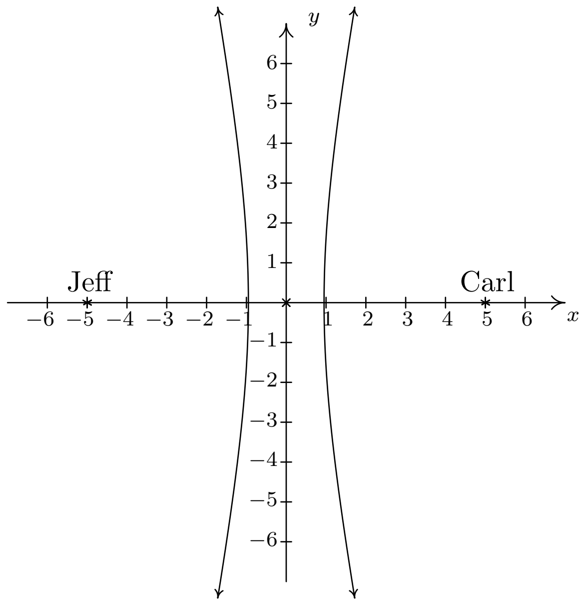
This means that Sasquatch is 1.9 miles closer to Jeff than it is to Carl. In other words, Sasquatch must lie on a path where

$$(\text{the distance to Carl}) - (\text{the distance to Jeff}) = 1.9$$

This is exactly the situation in the definition of a hyperbola, Definition 7.6. In this case, Jeff and Carl are located at the foci,[3] and our fixed distance d is 1.9. For simplicity, we assume the hyperbola is centered at $(0,0)$ with its foci at $(-5,0)$ and $(5,0)$. Schematically, we have

[2]GPS now rules the positioning kingdom. Is there still a place for LORAN and other land-based systems? Do satellites ever malfunction?

[3]We usually like to be the *center* of attention, but being the *focus* of attention works equally well.



We are seeking a curve of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the distance from the center to each focus is $c = 5$. As we saw in the derivation of the standard equation of the hyperbola, Equation 7.6, $d = 2a$, so that $2a = 1.9$, or $a = 0.95$ and $a^2 = 0.9025$. All that remains is to find b^2. To that end, we recall that $a^2 + b^2 = c^2$ so $b^2 = c^2 - a^2 = 25 - 0.9025 = 24.0975$. Since Sasquatch is closer to Jeff than it is to Carl, it must be on the western (left hand) branch of $\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1$. □

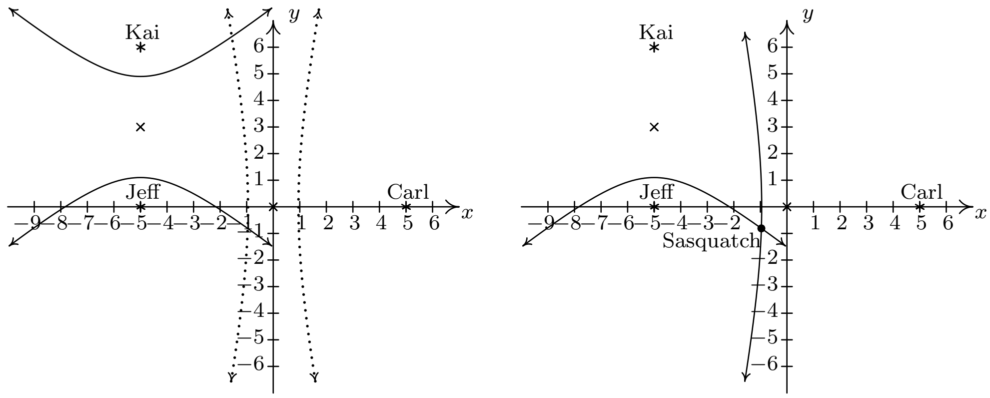
In our previous example, we did not have enough information to pin down the exact location of Sasquatch. To accomplish this, we would need a third observer.

Example 7.5.5. By a stroke of luck, Kai was also camping in the woods during the events of the previous example. He was located 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. Use this added information to locate Sasquatch.

Solution. Kai and Jeff are now the foci of a second hyperbola where the fixed distance d can be determined as before

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 18\,\text{seconds} = 3.8\,\text{miles}$$

Since Jeff was positioned at $(-5, 0)$, we place Kai at $(-5, 6)$. This puts the center of the new hyperbola at $(-5, 3)$. Plotting Kai's position and the new center gives us the diagram below on the left. The second hyperbola is vertical, so it must be of the form $\frac{(y-3)^2}{b^2} - \frac{(x+5)^2}{a^2} = 1$. As before, the distance d is the length of the major axis, which in this case is $2b$. We get $2b = 3.8$ so that $b = 1.9$ and $b^2 = 3.61$. With Kai 6 miles due North of Jeff, we have that the distance from the center to the focus is $c = 3$. Since $a^2 + b^2 = c^2$, we get $a^2 = c^2 - b^2 = 9 - 3.61 = 5.39$. Kai heard the Sasquatch call after Jeff, so Kai is farther from Sasquatch than Jeff. Thus Sasquatch must lie on the southern branch of the hyperbola $\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1$. Looking at the western branch of the hyperbola determined by Jeff and Carl along with the southern branch of the hyperbola determined by Kai and Jeff, we see that there is exactly one point in common, and this is where Sasquatch must have been when it called.



To determine the coordinates of this point of intersection exactly, we would need techniques for solving systems of non-linear equations (which we won't see until Section 8.7), so we use the calculator[4] Doing so, we get Sasquatch is approximately at $(-0.9629, -0.8113)$. □

Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ for different choices of A, C, D, E, and[5] F. While we've seen examples[6] demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[7]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Put the equation in the form of Equation 7.2 (if x is squared) or Equation 7.3 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and B.

- If $A = B$, the graph is a circle. Put the equation in the form of Equation 7.1.
- If $A \neq B$ but A and B have the *same sign*, the graph is an ellipse. Put the equation in the form of Equation 7.4.
- If A and B have the *different signs*, the graph is a hyperbola. Put the equation in the form of either Equation 7.6 or Equation 7.7.

[a]That is, a parabola, circle, ellipse, or hyperbola – see Section 7.1.

[4]First solve each hyperbola for y, and choose the correct equation (branch) before proceeding.
[5]See Section 11.6 to see why we skip B.
[6]Examples 7.2.3, 7.3.4, 7.4.3, and 7.5.3, in particular.
[7]We formalize this in Exercise 34.

7.5.1 Exercises

In Exercises 1 - 8, graph the hyperbola. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

9. $12x^2 - 3y^2 + 30y - 111 = 0$
10. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$
11. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$
12. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

In Exercises 13 - 18, find the standard form of the equation of the hyperbola which has the given properties.

13. Center $(3, 7)$, Vertex $(3, 3)$, Focus $(3, 2)$
14. Vertex $(0, 1)$, Vertex $(8, 1)$, Focus $(-3, 1)$
15. Foci $(0, \pm 8)$, Vertices $(0, \pm 5)$.
16. Foci $(\pm 5, 0)$, length of the Conjugate Axis 6
17. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Conjugate Axis $(8, 4)$, $(8, 0)$
18. Vertex $(-10, 5)$, Asymptotes $y = \pm\frac{1}{2}(x - 6) + 5$

In Exercises 19 - 28, find the standard form of the equation using the guidelines on page 540 and then graph the conic section.

19. $x^2 - 2x - 4y - 11 = 0$
20. $x^2 + y^2 - 8x + 4y + 11 = 0$
21. $9x^2 + 4y^2 - 36x + 24y + 36 = 0$
22. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

23. $y^2 + 8y - 4x + 16 = 0$

24. $4x^2 + y^2 - 8x + 4 = 0$

25. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

26. $x^2 + y^2 - 6x + 4y + 14 = 0$

27. $2x^2 + 4y^2 + 12x - 8y + 25 = 0$

28. $4x^2 - 5y^2 - 40x - 20y + 160 = 0$

29. The graph of a vertical or horizontal hyperbola clearly fails the Vertical Line Test, Theorem 1.1, so the equation of a vertical of horizontal hyperbola does not define y as a function of x.[8] However, much like with circles, horizontal parabolas and ellipses, we can split a hyperbola into pieces, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the hyperbolas given in Exercises 1 - 8 above. How many pieces do you need for a vertical hyperbola? How many for a horizontal hyperbola?

30. The location of an earthquake's epicenter − the point on the surface of the Earth directly above where the earthquake actually occurred − can be determined by a process similar to how we located Sasquatch in Example 7.5.5. (As we said back in Exercise 75 in Section 6.1, earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a course in Geology or the U.S. Geological Survey's Earthquake Hazards Program found here.) Our technique works only for relatively small distances because we need to assume that the Earth is flat in order to use hyperbolas in the plane.[9] The P-waves ("P" stands for Primary) of an earthquake in Sasquatchia travel at 6 kilometers per second.[10] Station A records the waves first. Then Station B, which is 100 kilometers due north of Station A, records the waves 2 seconds later. Station C, which is 150 kilometers due west of Station A records the waves 3 seconds after that (a total of 5 seconds after Station A). Where is the epicenter?

31. The notion of eccentricity introduced for ellipses in Definition 7.5 in Section 7.4 is the same for hyperbolas in that we can define the eccentricity e of a hyperbola as

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

   (a) With the help of your classmates, explain why $e > 1$ for any hyperbola.

   (b) Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and eccentricity $e = 2$.

   (c) With the help of your classmates, find the eccentricity of each of the hyperbolas in Exercises 1 - 8. What role does eccentricity play in the shape of the graphs?

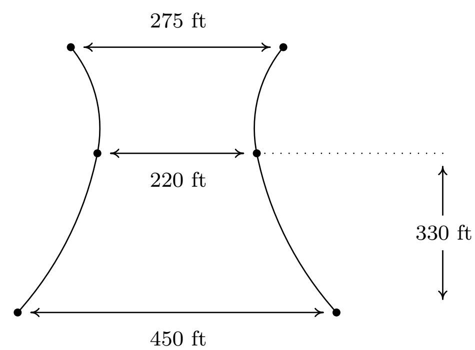
32. On page 510 in Section 7.3, we discussed paraboloids of revolution when studying the design of satellite dishes and parabolic mirrors. In much the same way, 'natural draft' cooling towers are often shaped as **hyperboloids of revolution**. Each vertical cross section of these towers

[8]We will see later in the text that the graphs of certain rotated hyperbolas pass the Vertical Line Test.

[9]Back in the Exercises in Section 1.1 you were asked to research people who believe the world is flat. What did you discover?

[10]Depending on the composition of the crust at a specific location, P-waves can travel between 5 kps and 8 kps.

is a hyperbola. Suppose the a natural draft cooling tower has the cross section below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.



33. With the help of your classmates, research the Cassegrain Telescope. It uses the reflective property of the hyperbola as well as that of the parabola to make an ingenious telescope.

34. With the help of your classmates show that if $Ax^2 + Cy^2 + Dx + Ey + F = 0$ determines a non-degenerate conic[11] then

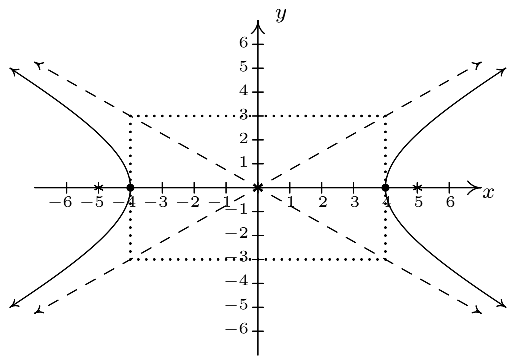
- $AC < 0$ means that the graph is a hyperbola
- $AC = 0$ means that the graph is a parabola
- $AC > 0$ means that the graph is an ellipse or circle

NOTE: This result will be generalized in Theorem 11.11 in Section 11.6.1.

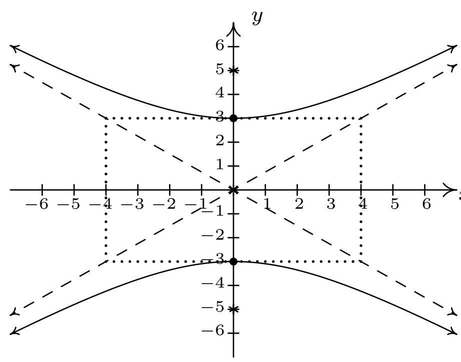
[11] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

7.5.2 Answers

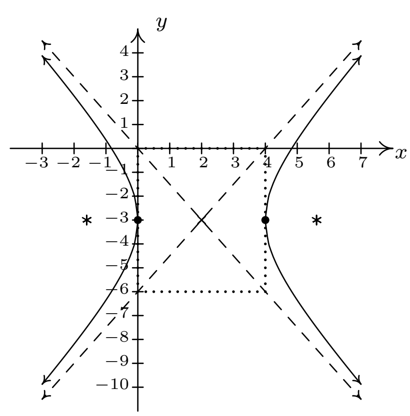
1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
   Center $(0,0)$
   Transverse axis on $y = 0$
   Conjugate axis on $x = 0$
   Vertices $(4,0), (-4,0)$
   Foci $(5,0), (-5,0)$
   Asymptotes $y = \pm\frac{3}{4}x$



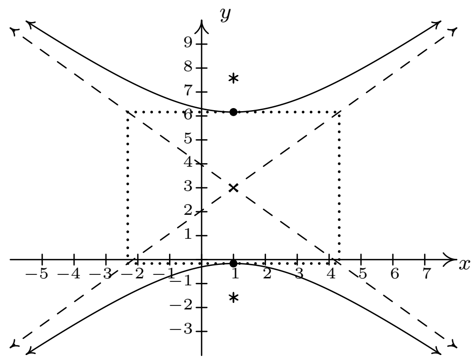
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
   Center $(0,0)$
   Transverse axis on $x = 0$
   Conjugate axis on $y = 0$
   Vertices $(0,3), (0,-3)$
   Foci $(0,5), (0,-5)$
   Asymptotes $y = \pm\frac{3}{4}x$



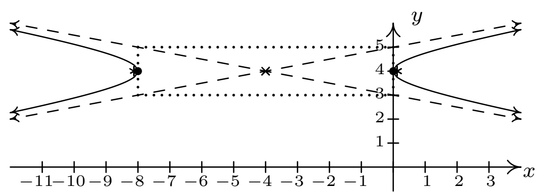
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Transverse axis on $y = -3$
   Conjugate axis on $x = 2$
   Vertices $(0,-3), (4,-3)$
   Foci $(2+\sqrt{13},-3), (2-\sqrt{13},-3)$
   Asymptotes $y = \pm\frac{3}{2}(x-2) - 3$



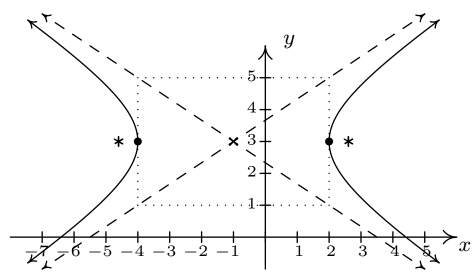
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
Center $(1,3)$
Transverse axis on $x=1$
Conjugate axis on $y=3$
Vertices $(1, 3+\sqrt{11}), (1, 3-\sqrt{11})$
Foci $(1, 3+\sqrt{21}), (1, 3-\sqrt{21})$
Asymptotes $y = \pm\frac{\sqrt{110}}{10}(x-1)+3$



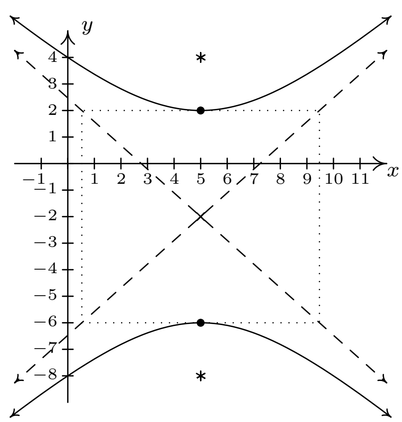
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
Center $(-4,4)$
Transverse axis on $y=4$
Conjugate axis on $x=-4$
Vertices $(-8,4), (0,4)$
Foci $(-4+\sqrt{17}, 4), (-4-\sqrt{17}, 4)$
Asymptotes $y = \pm\frac{1}{4}(x+4)+4$



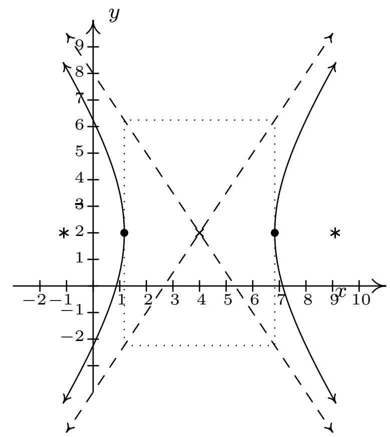
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
Center $(-1,3)$
Transverse axis on $y=3$
Conjugate axis on $x=-1$
Vertices $(2,3), (-4,3)$
Foci $\left(-1+\sqrt{13}, 3\right), \left(-1-\sqrt{13}, 3\right)$
Asymptotes $y = \pm\frac{2}{3}(x+1)+3$



7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
Center $(5,-2)$
Transverse axis on $x=5$
Conjugate axis on $y=-2$
Vertices $(5,2), (5,-6)$
Foci $(5,4), (5,-8)$
Asymptotes $y = \pm\frac{2\sqrt{5}}{5}(x-5)-2$



8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$
Center $(4,2)$
Transverse axis on $y = 2$
Conjugate axis on $x = 4$
Vertices $\left(4+2\sqrt{2},2\right), \left(4-2\sqrt{2},2\right)$
Foci $\left(4+\sqrt{26},2\right), \left(4-\sqrt{26},2\right)$
Asymptotes $y = \pm\frac{3}{2}(x-4)+2$



9. $\dfrac{x^2}{3} - \dfrac{(y-5)^2}{12} = 1$
Center $(0,5)$
Transverse axis on $y = 5$
Conjugate axis on $x = 0$
Vertices $(\sqrt{3},5), (-\sqrt{3},5)$
Foci $(\sqrt{15},5), (-\sqrt{15},5)$
Asymptotes $y = \pm 2x + 5$

10. $\dfrac{(y+2)^2}{5} - \dfrac{(x-3)^2}{18} = 1$
Center $(3,-2)$
Transverse axis on $x = 3$
Conjugate axis on $y = -2$
Vertices $(3,-2+\sqrt{5}), (3,-2-\sqrt{5})$
Foci $(3,-2+\sqrt{23}), (3,-2-\sqrt{23})$
Asymptotes $y = \pm\frac{\sqrt{10}}{6}(x-3)-2$

11. $\dfrac{(x-3)^2}{25} - \dfrac{(y+1)^2}{9} = 1$
Center $(3,-1)$
Transverse axis on $y = -1$
Conjugate axis on $x = 3$
Vertices $(8,-1), (-2,-1)$
Foci $\left(3+\sqrt{34},-1\right), \left(3-\sqrt{34},-1\right)$
Asymptotes $y = \pm\frac{3}{5}(x-3)-1$

12. $\dfrac{(y+4)^2}{6} - \dfrac{(x+2)^2}{5} = 1$
Center $(-2,-4)$
Transverse axis on $x = -2$
Conjugate axis on $y = -4$
Vertices $\left(-2,-4+\sqrt{6}\right), \left(-2,-4-\sqrt{6}\right)$
Foci $\left(-2,-4+\sqrt{11}\right), \left(-2,-4-\sqrt{11}\right)$
Asymptotes $y = \pm\frac{\sqrt{30}}{5}(x+2)-4$

13. $\dfrac{(y-7)^2}{16} - \dfrac{(x-3)^2}{9} = 1$

14. $\dfrac{(x-4)^2}{16} - \dfrac{(y-1)^2}{33} = 1$

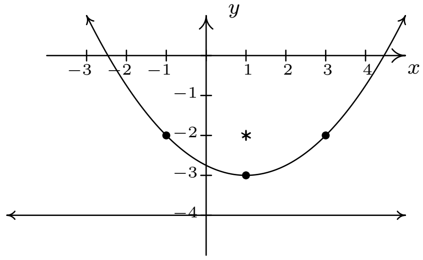
15. $\dfrac{y^2}{25} - \dfrac{x^2}{39} = 1$

16. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

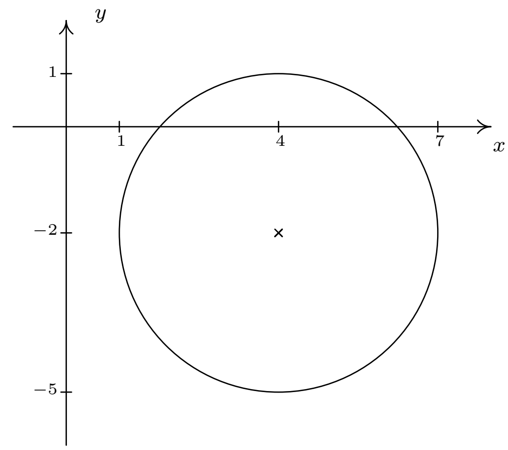
17. $\dfrac{(x-8)^2}{25} - \dfrac{(y-2)^2}{4} = 1$

18. $\dfrac{(x-6)^2}{256} - \dfrac{(y-5)^2}{64} = 1$

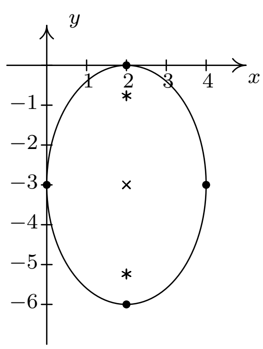
19. $(x-1)^2 = 4(y+3)$



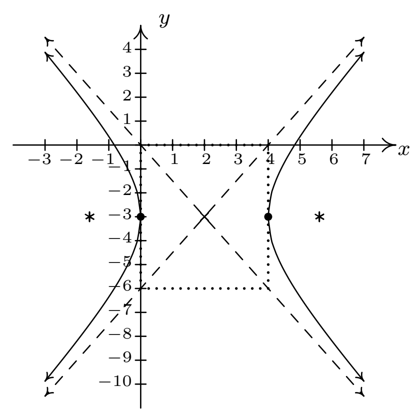
20. $(x-4)^2 + (y+2)^2 = 9$



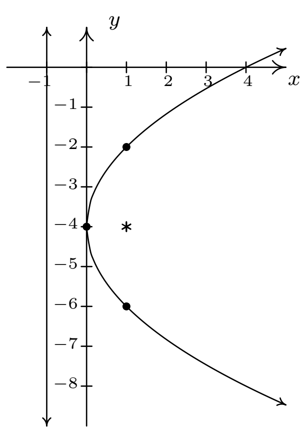
21. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$



22. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

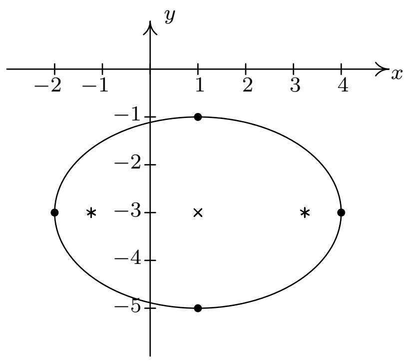


23. $(y+4)^2 = 4x$



24. $\dfrac{(x-1)^2}{1} + \dfrac{y^2}{4} = 0$
The graph is the point $(1, 0)$ only.

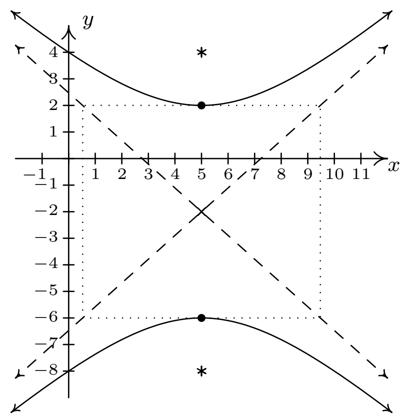
25. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$



26. $(x-3)^2 + (y+2)^2 = -1$
There is no graph.

27. $\dfrac{(x+3)^2}{2} + \dfrac{(y-1)^2}{1} = -\dfrac{3}{4}$
There is no graph.

28. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$



30. By placing Station A at $(0,-50)$ and Station B at $(0,50)$, the two second time difference yields the hyperbola $\frac{y^2}{36} - \frac{x^2}{2464} = 1$ with foci A and B and center $(0,0)$. Placing Station C at $(-150,-50)$ and using foci A and C gives us a center of $(-75,-50)$ and the hyperbola $\frac{(x+75)^2}{225} - \frac{(y+50)^2}{5400} = 1$. The point of intersection of these two hyperbolas which is closer to A than B and closer to A than C is $(-57.8444, -9.21336)$ so that is the epicenter.

31. (b) $\dfrac{x^2}{9} - \dfrac{y^2}{27} = 1$.

32. The tower may be modeled (approximately)[12] by $\frac{x^2}{12100} - \frac{(y-330)^2}{34203} = 1$. To find the height, we plug in $x = 137.5$ which yields $y \approx 191$ or $y \approx 469$. Since the top of the tower is above the narrowest point, we get the tower is approximately 469 feet tall.

[12] The exact value underneath $(y-330)^2$ is $\frac{52707600}{1541}$ in case you need more precision.

Chapter 8

Systems of Equations and Matrices

8.1 Systems of Linear Equations: Gaussian Elimination

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The x-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the y-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to x and y are of interest, we have what is known as a **system of equations**, usually written as

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ y & = & g(x) \end{array} \right.$$

The 'curly bracket' notation means we are to find all **pairs** of points (x, y) which satisfy **both** equations. We begin our study of systems of equations by reviewing some basic notions from Intermediate Algebra.

Definition 8.1. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

For reasons which will become clear later in the section, we are using subscripts in Definition 8.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables[1] by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$. If a_1 and a_2 are both 0, then depending on c, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play

[1] Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. We leave it to the reader to explain why these do not satisfy Definition 8.1. From what we know from Sections 1.2 and 2.1, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example reviews some of the basic techniques first learned in Intermediate Algebra.

Example 8.1.1. Solve the following systems of equations. Check your answer algebraically and graphically.

1. $\left\{ \begin{array}{rcr} 2x - y & = & 1 \\ y & = & 3 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 3x + 4y & = & -2 \\ -3x - y & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$
4. $\left\{ \begin{array}{rcr} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$
5. $\left\{ \begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array} \right.$
6. $\left\{ \begin{array}{rcr} x - y & = & 0 \\ x + y & = & 2 \\ -2x + y & = & -2 \end{array} \right.$

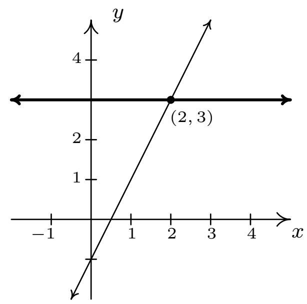
Solution.

1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of x, we **substitute** this value for y into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

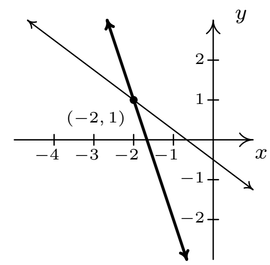
2. To solve the second system, we use the **addition** method to **eliminate** the variable x. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.



$2x - y = 1$
$\mathbf{y = 3}$



$3x + 4y = -2$
$\mathbf{-3x - y = 5}$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\left\{ \begin{array}{rcr} 5x - 12y & = & 21 \\ 4x + 6y & = & 9 \end{array} \right.$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by -5, we will be in a position to eliminate the x term

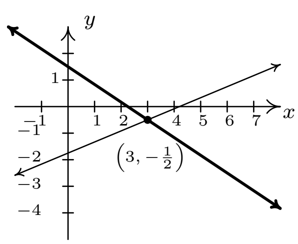
$$\begin{array}{rrcr} & 20x - 48y & = & 84 \\ + & (-20x - 30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice − in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.

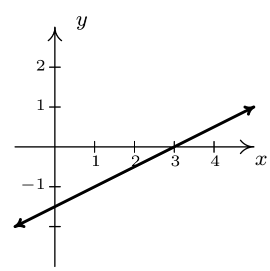
4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and the both sides of the second equation by -2, we are ready to eliminate the x

$$\begin{array}{rrcr} & 6x-12y & = & 18 \\ + & (-6x+12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

We eliminated not only the x, but the y as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x, y) satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$. One way to describe the solution set to this system is to use the roster method[2] and write $\{(x, y) \,|\, 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system**. Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of x, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding y-value of a solution. Since we have no restriction on x, it is called a **free variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t, \frac{1}{2}t - \frac{3}{2}\right) \mid -\infty < t < \infty\right\}$.[3] For specific values of t, we can generate solutions. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$; $t = 117$ gives us $(117, 57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always. We claim that for any real number t, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too. Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.



$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$
$\boldsymbol{\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}}$



$2x - 4y = 6$
$\boldsymbol{3x - 6y = 9}$
(Same line.)

[2]See Section 1.2 for a review of this.

[3]Note that we could have just as easily chosen to solve $2x - 4y = 6$ for x to obtain $x = 2y + 3$. Letting y be the parameter t, we have that for any value of t, $x = 2t + 3$, which gives $\{(2t + 3, t) \,|\, -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by -3, we set the stage to eliminate x

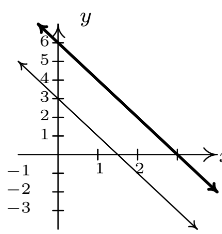
$$\begin{array}{rrcr} & 12x+6y & = & 18 \\ + & (-12x-6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

As in the previous example, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0=-18$. This tells us that it is impossible to find a pair (x, y) which satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x+3y=9$ and $4x+2y=12$ are distinct and parallel, so they do not intersect.

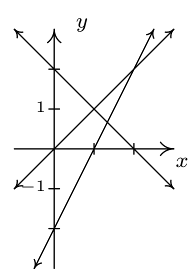
6. We can begin to solve our last system by adding the first two equations

$$\begin{array}{rrcr} & x-y & = & 0 \\ + & (x+y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x=1$. Substituting this into the first equation gives $1-y=0$ so that $y=1$. We seem to have determined a solution to our system, $(1,1)$. While this checks in the first two equations, when we substitute $x=1$ and $y=1$ into the third equation, we get $-2(1)+(1)=-2$ which simplifies to the contradiction $-1=-2$. Graphing the lines $x-y=0$, $x+y=2$, and $-2x+y=-2$, we see that the first two lines do, in fact, intersect at $(1,1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.



$6x+3y=9$
$\mathbf{4x+2y=12}$



$y-x=0$
$y+x=2$
$-2x+y=-2$

□

A few remarks about Example 8.1.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[4] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 8.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

In order to move this section beyond a review of Intermediate Algebra, we now define what is meant by a linear equation in n variables.

Definition 8.2. A **linear equation in n variables**, x_1, x_2, ..., x_n is an equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ where a_1, a_2, $\ldots a_n$ and c are real numbers and at least one of a_1, a_2, ..., a_n is nonzero.

Instead of using more familiar variables like x, y, and even z and/or w in Definition 8.2, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.) As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables x_1 and x_2 as the equation $3x - y = 4$ does between the variables x and y. In addition, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$. Coupling more than one linear equation in n variables results in a **system of linear equations in n variables**. When solving these systems, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations we've already employed. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[9]

[4] In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5] The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[6] If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7] We need more than two variables to give an example of the latter.

[8] Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

[9] That is, a system with the same solution set.

Theorem 8.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

We have seen plenty of instances of the second and third moves in Theorem 8.1 when we solved the systems Example 8.1.1. The first move, while it obviously admits an equivalent system, seems silly. Our perception will change as we consider more equations and more variables in this, and later sections.

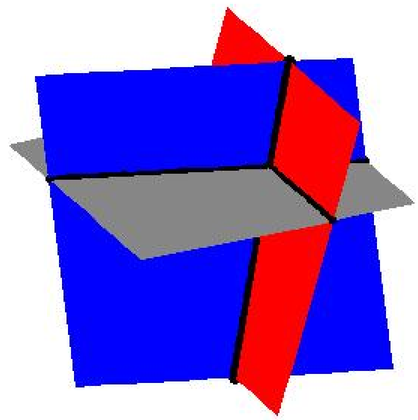
Consider the system of equations

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

Clearly $z = -1$, and we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Finally, we substitute $y = \frac{7}{2}$ and $z = -1$ into the first equation to get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$, so that $x = \frac{8}{3}$. The reader can verify that these values of x, y and z satisfy all three original equations. It is tempting for us to write the solution to this system by extending the usual (x, y) notation to (x, y, z) and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[10] Moreover, just as equations involving the variables x and y describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables x, y, and z describe objects called **surfaces** in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[11] so our intersection point is where all three of these lines meet.

[10]You were asked to think about this in Exercise 40 in Section 1.1.

[11]In fact, these lines are described by the parametric solutions to the systems formed by taking any two of these equations by themselves.



Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

we note the reason it was so easy to solve is that the third equation is solved for z, the second equation involves only y and z, and since the coefficient of y is 1, it makes it easy to solve for y using our known value for z. Lastly, the coefficient of x in the first equation is 1 making it easy to substitute the known values of y and z and then solve for x. We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the **leading variable** of that equation.

Definition 8.3. A system of linear equations with variables x_1, x_2, $\ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.
2. The leading variable in each equation has coefficient 1.
3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.
4. Any equation without variables[a] cannot be placed above an equation with variables.

[a]necessarily an identity or contradiction

In our previous system, if we make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[12] An example of a more complicated system in triangular form is

$$\left\{ \begin{array}{rcr} x_1 - 4x_3 + x_4 - x_6 & = & 6 \\ x_2 + 2x_3 & = & 1 \\ x_4 + 3x_5 - x_6 & = & 8 \\ x_5 + 9x_6 & = & 10 \end{array} \right.$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 8.1.

Example 8.1.2. Use Theorem 8.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

1. $\left\{ \begin{array}{rcr} 3x - y + z & = & 3 \\ 2x - 4y + 3z & = & 16 \\ x - y + z & = & 5 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 2x + 3y - z & = & 1 \\ 10x - z & = & 2 \\ 4x - 9y + 2z & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} 3x_1 + x_2 + x_4 & = & 6 \\ 2x_1 + x_2 - x_3 & = & 4 \\ x_2 - 3x_3 - 2x_4 & = & 0 \end{array} \right.$

Solution.

1. For definitiveness, we label the topmost equation in the system $E1$, the equation beneath that $E2$, and so forth. We now attempt to put the system in triangular form using an algorithm known as **Gaussian Elimination**. What this means is that, starting with x, we transform the system so that conditions 2 and 3 in Definition 8.3 are satisfied. Then we move on to the next variable, in this case y, and repeat. Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an x in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 8.1 and interchange $E1$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & 3x - y + z & = & 3 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & x - y + z & = & 5 \end{array} \right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right.$$

To satisfy Definition 8.3, we need to eliminate the x's from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the x from $E2$, we need to multiply $E1$ by -2 then add; to eliminate the x from $E3$, we need to multiply $E1$ by -3 then add. Applying the third move listed in Theorem 8.1 twice, we get

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -3E1 + E3]{\text{Replace } E2 \text{ with } -2E1 + E2} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & -2y + z & = & 6 \\ (E3) & 2y - 2z & = & -12 \end{array} \right.$$

[12] If letters are used instead of subscripted variables, Definition 8.3 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

Now we enforce the conditions stated in Definition 8.3 for the variable y. To that end we need to get the coefficient of y in $E2$ equal to 1. We apply the second move listed in Theorem 8.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & -2y+z & = & 6 \\ (E3) & 2y-2z & = & -12 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array}\right.$$

To eliminate the y in $E3$, we add $-2E2$ to it.

$$\left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } -2E2+E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array}\right.$$

Finally, we apply the second move from Theorem 8.1 one last time and multiply $E3$ by -1 to satisfy the conditions of Definition 8.3 for the variable z.

$$\left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & z & = & 6 \end{array}\right.$$

Now we proceed to substitute. Plugging in $z=6$ into $E2$ gives $y-3=-3$ so that $y=0$. With $y=0$ and $z=6$, $E1$ becomes $x-0+6=5$, or $x=-1$. Our solution is $(-1,0,6)$. We leave it to the reader to check that substituting the respective values for x, y, and z into the original system results in three identities. Since we have found a solution, the system is consistent; since there are no free variables, it is independent.

2. Proceeding as we did in 1, our first step is to get an equation with x in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\left\{\begin{array}{lrcr} (E1) & 2x+3y-z & = & 1 \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array}\right.$$

Now it's time to take care of the x's in $E2$ and $E3$.

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array}\right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1+E3]{\text{Replace } E2 \text{ with } -10E1+E2} \left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

Our next step is to get the coefficient of y in $E2$ equal to 1. To that end, we have

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y-\frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

Finally, we rid $E3$ of y.

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y-\frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y+4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } 15E2+E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 0 & = & 6 \end{array}\right.$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 8.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on x_1.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1+x_2+x_4 & = & 6 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right.$$

Next we eliminate x_1 from $E2$

$$\left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -2E1+E2]{\text{Replace } E2} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 & = & 0 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right.$$

We switch $E2$ and $E3$ to get a coefficient of 1 for x_2.

$$\left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 & = & 0 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Switch } E2 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & x_2-3x_3-2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 & = & 0 \end{array}\right.$$

Finally, we eliminate x_2 in $E3$.

$$\left\{ \begin{array}{rrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array} \right. \xrightarrow[\text{with } -\frac{1}{3}E2+E3]{\text{Replace } E3} \left\{ \begin{array}{rrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array} \right.$$

Equation $E3$ reduces to $0 = 0$,which is always true. Since we have no equations with x_3 or x_4 as leading variables, they are both free, which means we have a consistent dependent system. We parametrize the solution set by letting $x_3 = s$ and $x_4 = t$ and obtain from $E2$ that $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2 - s - t$. Our solution is the set $\{(2 - s - t, 2s + 3t, s, t) \mid -\infty < s, t < \infty\}$.[13] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system. □

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving 2 above, it is clear after we eliminated the x's in the second step to get the system

$$\left\{ \begin{array}{rrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

that equations $E2$ and $E3$ when taken together form a contradiction since we have identical left hand sides and different right hand sides. The algorithm takes two more steps to reach this contradiction. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, thus it gets called **back-substitution**. This may also be inefficient in many cases. Rest assured, the technique of substitution as you may have learned it in Intermediate Algebra will once again take center stage in Section 8.7. Lastly, we note that the system in 3 above is underdetermined, and as it is consistent, we have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations.

Example 8.1.3. Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

Solution. We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call x), the amount (in mL) of the 90% stock solution (which we'll call y) and the amount (in mL) of water (which we'll call w). We now need to determine some relationships between these variables. Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each

[13] Here, any choice of s and t will determine a solution which is a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions and the water. That is

$$\text{amount of 30\% stock solution} + \text{amount of 90\% stock solution} + \text{amount of water} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x + y + w = 500$. Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 90\% stock solution} = 200\,\text{mL}$$

The amount of acid in x mL of 30% stock is $0.30x$ and the amount of acid in y mL of 90% solution is $0.90y$. We have $0.30x + 0.90y = 200$. Converting to fractions,[14] our system of equations becomes

$$\left\{ \begin{array}{rcl} x + y + w &=& 500 \\ \frac{3}{10}x + \frac{9}{10}y &=& 200 \end{array} \right.$$

We first eliminate the x from the second equation

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w &=& 500 \\ (E2) & \frac{3}{10}x + \frac{9}{10}y &=& 200 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1 + E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w &=& 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w &=& 50 \end{array} \right.$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w &=& 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w &=& 50 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w &=& 500 \\ (E2) & y - \frac{1}{2}w &=& \frac{250}{3} \end{array} \right.$$

Notice that we have no equation to determine w, and as such, w is free. We set $w = t$ and from $E2$ get $y = \frac{1}{2}t + \frac{250}{3}$. Substituting into $E1$ gives $x + \left(\frac{1}{2}t + \frac{250}{3}\right) + t = 500$ so that $x = -\frac{3}{2}t + \frac{1250}{3}$. This system is consistent, dependent and its solution set is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid -\infty < t < \infty\right\}$. While this answer checks algebraically, we have neglected to take into account that x, y and w, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$. The constraint $x \geq 0$ gives us $-\frac{3}{2}t + \frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t + \frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. Our final answer is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid 0 \leq t \leq \frac{2500}{9}\right\}$. Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y = 100$ so that $\frac{1}{2}t + \frac{250}{3} = 100$, and we get $t = \frac{100}{3}$. This means the amount of 30% solution required is $x = -\frac{3}{2}t + \frac{1250}{3} = -\frac{3}{2}\left(\frac{100}{3}\right) + \frac{1250}{3} = \frac{1100}{3}$ mL, and for the water, $w = t = \frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. $\square$

[14] We do this only because we believe students can use all of the practice with fractions they can get!

8.1.1 EXERCISES

(Review Exercises) In Exercises 1 - 8, take a trip down memory lane and solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent. Check your answers both algebraically and graphically.

1. $\left\{ \begin{array}{rcr} x + 2y & = & 5 \\ x & = & 6 \end{array} \right.$

2. $\left\{ \begin{array}{rcr} 2y - 3x & = & 1 \\ y & = & -3 \end{array} \right.$

3. $\left\{ \begin{array}{rcr} \frac{x+2y}{4} & = & -5 \\ \frac{3x-y}{2} & = & 1 \end{array} \right.$

4. $\left\{ \begin{array}{rcr} \frac{2}{3}x - \frac{1}{5}y & = & 3 \\ \frac{1}{2}x + \frac{3}{4}y & = & 1 \end{array} \right.$

5. $\left\{ \begin{array}{rcr} \frac{1}{2}x - \frac{1}{3}y & = & -1 \\ 2y - 3x & = & 6 \end{array} \right.$

6. $\left\{ \begin{array}{rcr} x + 4y & = & 6 \\ \frac{1}{12}x + \frac{1}{3}y & = & \frac{1}{2} \end{array} \right.$

7. $\left\{ \begin{array}{rcr} 3y - \frac{3}{2}x & = & -\frac{15}{2} \\ \frac{1}{2}x - y & = & \frac{3}{2} \end{array} \right.$

8. $\left\{ \begin{array}{rcr} \frac{5}{6}x + \frac{5}{3}y & = & -\frac{7}{3} \\ -\frac{10}{3}x - \frac{20}{3}y & = & 10 \end{array} \right.$

In Exercises 9 - 26, put each system of linear equations into triangular form and solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

9. $\left\{ \begin{array}{rcr} -5x + y & = & 17 \\ x + y & = & 5 \end{array} \right.$

10. $\left\{ \begin{array}{rcr} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

11. $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{array} \right.$

12. $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 6 \end{array} \right.$

13. $\left\{ \begin{array}{rcr} x + y + z & = & -17 \\ y - 3z & = & 0 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{array} \right.$

15. $\left\{ \begin{array}{rcr} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{array} \right.$

17. $\left\{ \begin{array}{rcr} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{array} \right.$

18. $\left\{ \begin{array}{rcr} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{array} \right.$

19. $\left\{ \begin{array}{rcr} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array} \right.$

20. $\left\{ \begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array} \right.$

21. $\begin{cases} x+y+z = 4 \\ 2x-4y-z = -1 \\ x-y = 2 \end{cases}$

22. $\begin{cases} x-y+z = 8 \\ 3x+3y-9z = -6 \\ 7x-2y+5z = 39 \end{cases}$

23. $\begin{cases} 2x-3y+z = -1 \\ 4x-4y+4z = -13 \\ 6x-5y+7z = -25 \end{cases}$

24. $\begin{cases} 2x_1+x_2-12x_3-x_4 = 16 \\ -x_1+x_2+12x_3-4x_4 = -5 \\ 3x_1+2x_2-16x_3-3x_4 = 25 \\ x_1+2x_2-5x_4 = 11 \end{cases}$

25. $\begin{cases} x_1-x_3 = -2 \\ 2x_2-x_4 = 0 \\ x_1-2x_2+x_3 = 0 \\ -x_3+x_4 = 1 \end{cases}$

26. $\begin{cases} x_1-x_2-5x_3+3x_4 = -1 \\ x_1+x_2+5x_3-3x_4 = 0 \\ x_2+5x_3-3x_4 = 1 \\ x_1-2x_2-10x_3+6x_4 = -1 \end{cases}$

27. Find two other forms of the parametric solution to Exercise 11 above by reorganizing the equations so that x or y can be the free variable.

28. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

29. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

30. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

31. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each are needed?

32. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

33. Discuss with your classmates how you would approach Exercise 32 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

34. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like? Explain.

8.1.2 Answers

1. Consistent independent
   Solution $\left(6, -\frac{1}{2}\right)$

2. Consistent independent
   Solution $\left(-\frac{7}{3}, -3\right)$

3. Consistent independent
   Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$

4. Consistent independent
   Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$

5. Consistent dependent
   Solution $\left(t, \frac{3}{2}t + 3\right)$
   for all real numbers t

6. Consistent dependent
   Solution $(6 - 4t, t)$
   for all real numbers t

7. Inconsistent
   No solution

8. Inconsistent
   No solution

Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

9. $\left\{ \begin{array}{rcr} x + y & = & 5 \\ y & = & 7 \end{array} \right.$

   Consistent independent
   Solution $(-2, 7)$

10. $\left\{ \begin{array}{rcr} x - \frac{5}{3}y - \frac{7}{3}z & = & -\frac{7}{3} \\ y + \frac{5}{4}z & = & 2 \\ z & = & 0 \end{array} \right.$

    Consistent independent
    Solution $(1, 2, 0)$

11. $\left\{ \begin{array}{rcr} x - \frac{1}{4}y + \frac{1}{4}z & = & \frac{5}{4} \\ y + 3z & = & 15 \\ 0 & = & 0 \end{array} \right.$

    Consistent dependent
    Solution $(-t + 5, -3t + 15, t)$
    for all real numbers t

12. $\left\{ \begin{array}{rcr} x - \frac{1}{4}y + \frac{1}{4}z & = & \frac{5}{4} \\ y + 3z & = & 15 \\ 0 & = & 1 \end{array} \right.$

    Inconsistent
    No solution

13. $\left\{ \begin{array}{rcr} x + y + z & = & -17 \\ y - 3z & = & 0 \end{array} \right.$

    Consistent dependent
    Solution $(-4t - 17, 3t, t)$
    for all real numbers t

14. $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ y - \frac{11}{5}z & = & -\frac{16}{5} \\ z & = & 1 \end{array} \right.$

    Consistent independent
    Solution $(2, -1, 1)$

15. $\left\{\begin{array}{rcr} x+y+2z & = & 0 \\ y-\frac{3}{2}z & = & 6 \\ z & = & -2 \end{array}\right.$

Consistent independent
Solution $(1,3,-2)$

16. $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z & = & -\frac{1}{2} \\ y+\frac{3}{5}z & = & \frac{3}{5} \\ 0 & = & 1 \end{array}\right.$

Inconsistent
no solution

17. $\left\{\begin{array}{rcr} x-y+z & = & -4 \\ y-7z & = & 17 \\ z & = & -2 \end{array}\right.$

Consistent independent
Solution $(1,3,-2)$

18. $\left\{\begin{array}{rcr} x-2y+2z & = & -2 \\ y & = & \frac{1}{2} \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $\left(-3,\frac{1}{2},1\right)$

19. $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z & = & \frac{1}{2} \\ y-\frac{2}{3}z & = & 0 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $\left(\frac{1}{3},\frac{2}{3},1\right)$

20. $\left\{\begin{array}{rcr} x-3y-4z & = & 3 \\ y+\frac{11}{13}z & = & \frac{4}{13} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(\frac{19}{13}t+\frac{51}{13},-\frac{11}{13}t+\frac{4}{13},t\right)$
for all real numbers t

21. $\left\{\begin{array}{rcr} x+y+z & = & 4 \\ y+\frac{1}{2}z & = & \frac{3}{2} \\ 0 & = & 1 \end{array}\right.$

Inconsistent
no solution

22. $\left\{\begin{array}{rcr} x-y+z & = & 8 \\ y-2z & = & -5 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $(4,-3,1)$

23. $\left\{\begin{array}{rcr} x-\frac{3}{2}y+\frac{1}{2}z & = & -\frac{1}{2} \\ y+z & = & -\frac{11}{2} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(-2t-\frac{35}{4},-t-\frac{11}{2},t\right)$
for all real numbers t

24. $\left\{\begin{array}{rcr} x_1+\frac{2}{3}x_2-\frac{16}{3}x_3-x_4 & = & \frac{25}{3} \\ x_2+4x_3-3x_4 & = & 2 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $(8s-t+7,-4s+3t+2,s,t)$
for all real numbers s and t

25. $\left\{\begin{array}{rcr} x_1-x_3 & = & -2 \\ x_2-\frac{1}{2}x_4 & = & 0 \\ x_3-\frac{1}{2}x_4 & = & 1 \\ x_4 & = & 4 \end{array}\right.$

Consistent independent
Solution $(1,2,3,4)$

26. $\left\{ \begin{array}{rcr} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_2 + 5x_3 - 3x_4 & = & \frac{1}{2} \\ 0 & = & 1 \\ 0 & = & 0 \end{array} \right.$ Inconsistent
No solution

27. If x is the free variable then the solution is $(t, 3t, -t + 5)$ and if y is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t + 5\right)$.

28. 13 chose the basic buffet and 14 chose the deluxe buffet.

29. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.

30. Skippy needs to invest \$6000 in the 3% account and \$4000 in the 8% account.

31. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.

32. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and t pounds of Type III where $0 \leq t \leq \frac{4}{3}$.

8.2 Systems of Linear Equations: Augmented Matrices

In Section 8.1 we introduced Gaussian Elimination as a means of transforming a system of linear equations into triangular form with the ultimate goal of producing an equivalent system of linear equations which is easier to solve. If we take a step back and study the process, we see that all of our moves are determined entirely by the *coefficients* of the variables involved, and not the variables themselves. Much the same thing happened when we studied long division in Section 3.2. Just as we developed synthetic division to streamline that process, in this section, we introduce a similar bookkeeping device to help us solve systems of linear equations. To that end, we define a **matrix** as a rectangular array of real numbers. We typically enclose matrices with square brackets, '[' and ']', and we size matrices by the number of rows and columns they have. For example, the **size** (sometimes called the **dimension**) of

$$\left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

is 2×3 because it has 2 rows and 3 columns. The individual numbers in a matrix are called its **entries** and are usually labeled with double subscripts: the first tells which row the element is in and the second tells which column it is in. The rows are numbered from top to bottom and the columns are numbered from left to right. Matrices themselves are usually denoted by uppercase letters (A, B, C, etc.) while their entries are usually denoted by the corresponding letter. So, for instance, if we have

$$A = \left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

then $a_{11} = 3$, $a_{12} = 0$, $a_{13} = -1$, $a_{21} = 2$, $a_{22} = -5$, and $a_{23} = 10$. We shall explore matrices as mathematical objects with their own algebra in Section 8.3 and introduce them here solely as a bookkeeping device. Consider the system of linear equations from number 2 in Example 8.1.2

$$\left\{ \begin{array}{lrcl} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right.$$

We encode this system into a matrix by assigning each equation to a corresponding row. Within that row, each variable and the constant gets its own column, and to separate the variables on the left hand side of the equation from the constants on the right hand side, we use a vertical bar, $|$. Note that in $E2$, since y is not present, we record its coefficient as 0. The matrix associated with this system is

$$\begin{array}{c} \\ (E1) \rightarrow \\ (E2) \rightarrow \\ (E3) \rightarrow \end{array} \begin{array}{c} \begin{array}{rrrr} x & y & z & c \end{array} \\ \left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \end{array}$$

This matrix is called an **augmented matrix** because the column containing the constants is appended to the matrix containing the coefficients.[1] To solve this system, we can use the same kind operations on the *rows* of the matrix that we performed on the *equations* of the system. More specifically, we have the following analog of Theorem 8.1 below.

Theorem 8.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b] Where we add entries in corresponding columns.

As a demonstration of the moves in Theorem 8.2, we revisit some of the steps that were used in solving the systems of linear equations in Example 8.1.2 of Section 8.1. The reader is encouraged to perform the indicated operations on the rows of the augmented matrix to see that the machinations are identical to what is done to the coefficients of the variables in the equations. We first see a demonstration of switching two rows using the first step of part 1 in Example 8.1.2.

$$\left\{\begin{array}{lrcr} (E1) & 3x-y+z & = & 3 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & x-y+z & = & 5 \end{array}\right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & 3x-y+z & = & 3 \end{array}\right.$$

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R3} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

Next, we have a demonstration of replacing a row with a nonzero multiple of itself using the first step of part 3 in Example 8.1.2.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1+x_2+x_4 & = & 6 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right.$$

$$\left[\begin{array}{rrrr|r} 3 & 1 & 0 & 1 & 6 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 2 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right]$$

Finally, we have an example of replacing a row with itself plus a multiple of another row using the second step from part 2 in Example 8.1.2.

[1] We shall study the coefficient and constant matrices separately in Section 8.3.

$$\left\{\begin{array}{lrcr}(E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2}\\ (E2) & 10x-z & = & 2\\ (E3) & 4x-9y+2z & = & 5\end{array}\right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1+E3]{\text{Replace } E2 \text{ with } -10E1+E2} \left\{\begin{array}{lrcr}(E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2}\\ (E2) & -15y+4z & = & -3\\ (E3) & -15y+4z & = & 3\end{array}\right.$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2}\\ 10 & 0 & -1 & 2\\ 4 & -9 & 2 & 5\end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -4R1+R3]{\text{Replace } R2 \text{ with } -10R1+R2} \left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2}\\ 0 & -15 & 4 & -3\\ 0 & -15 & 4 & 3\end{array}\right]$$

The matrix equivalent of 'triangular form' is **row echelon form**. The reader is encouraged to refer to Definition 8.3 for comparison. Note that the analog of 'leading variable' of an equation is 'leading entry' of a row. Specifically, the first nonzero entry (if it exists) in a row is called the **leading entry** of that row.

Definition 8.4. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

To solve a system of a linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to the rows to get the matrix into row-echelon form. We then decode the matrix and back substitute. The next example illustrates this nicely.

Example 8.2.1. Use an augmented matrix to transform the following system of linear equations into triangular form. Solve the system.

$$\left\{\begin{array}{rcl} 3x-y+z & = & 8\\ x+2y-z & = & 4\\ 2x+3y-4z & = & 10\end{array}\right.$$

Solution. We first encode the system into an augmented matrix.

$$\left\{\begin{array}{rcl} 3x-y+z & = & 8\\ x+2y-z & = & 4\\ 2x+3y-4z & = & 10\end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 3 & -1 & 1 & 8\\ 1 & 2 & -1 & 4\\ 2 & 3 & -4 & 10\end{array}\right]$$

Thinking back to Gaussian Elimination at an equations level, our first order of business is to get x in $E1$ with a coefficient of 1. At the matrix level, this means getting a leading 1 in $R1$. This is in accordance with the first criteria in Definition 8.4. To that end, we interchange $R1$ and $R2$.

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8\\ 1 & 2 & -1 & 4\\ 2 & 3 & -4 & 10\end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4\\ 3 & -1 & 1 & 8\\ 2 & 3 & -4 & 10\end{array}\right]$$

Our next step is to eliminate the x's from $E2$ and $E3$. From a matrix standpoint, this means we need 0's below the leading 1 in $R1$. This guarantees the leading 1 in $R2$ will be to the right of the leading 1 in $R1$ in accordance with the second requirement of Definition 8.4.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -2R1+R3]{\text{Replace } R2 \text{ with } -3R1+R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right]$$

Now we repeat the above process for the variable y which means we need to get the leading entry in $R2$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -\frac{1}{7}R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right]$$

To guarantee the leading 1 in $R3$ is to the right of the leading 1 in $R2$, we get a 0 in the second column of $R3$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } R2+R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right]$$

Finally, we get the leading entry in $R3$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{7}{18}R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Decoding from the matrix gives a system in triangular form

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x+2y-z & = & 4 \\ y-\frac{4}{7}z & = & \frac{4}{7} \\ z & = & -1 \end{array}\right.$$

We get $z=-1$, $y=\frac{4}{7}z+\frac{4}{7}=\frac{4}{7}(-1)+\frac{4}{7}=0$ and $x=-2y+z+4=-2(0)+(-1)+4=3$ for a final answer of $(3,0,-1)$. We leave it to the reader to check. □

As part of Gaussian Elimination, we used row operations to obtain 0's beneath each leading 1 to put the matrix into row echelon form. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

Definition 8.5. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1s are the only nonzero entry in their respective columns.

Of what significance is the reduced row echelon form of a matrix? To illustrate, let's take the row echelon form from Example 8.2.1 and perform the necessary steps to put into reduced row echelon form. We start by using the leading 1 in $R3$ to zero out the numbers in the rows above it.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\text{Replace } R2 \text{ with } \frac{4}{7}R3+R2]{\text{Replace } R1 \text{ with } R3+R1} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Finally, we take care of the 2 in $R1$ above the leading 1 in $R2$.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -2R2+R1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To our surprise and delight, when we decode this matrix, we obtain the solution instantly without having to deal with any back-substitution at all.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x & = & 3 \\ y & = & 0 \\ z & = & -1 \end{array}\right.$$

Note that in the previous discussion, we could have started with $R2$ and used it to get a zero above its leading 1 and then done the same for the leading 1 in $R3$. By starting with $R3$, however, we get more zeros first, and the more zeros there are, the faster the remaining calculations will be.[2] It is also worth noting that while a matrix has several[3] row echelon forms, it has only one reduced row echelon form. The process by which we have put a matrix into reduced row echelon form is called **Gauss-Jordan Elimination**.

Example 8.2.2. Solve the following system using an augmented matrix. Use Gauss-Jordan Elimination to put the augmented matrix into reduced row echelon form.

$$\left\{\begin{array}{rcr} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right.$$

Solution. We first encode the system into a matrix. (Pay attention to the subscripts!)

$$\left\{\begin{array}{rcr} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

Next, we get a leading 1 in the first column of $R1$.

$$\left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -\frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

[2]Carl also finds starting with $R3$ to be more symmetric, in a purely poetic way.

[3]infinite, in fact

Now we eliminate the nonzero entry below our leading 1.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -2R1+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We proceed to get a leading 1 in $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } \frac{3}{2}R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We now zero out the entry below the leading 1 in $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -4R2+R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right]$$

Next, it's time for a leading 1 in $R3$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{1}{24}R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

The matrix is now in row echelon form. To get the reduced row echelon form, we start with the last leading 1 we produced and work to get 0's above it.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -6R3+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

Lastly, we get a 0 above the leading 1 of $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R2+R1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

At last, we decode to get

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{ \begin{array}{rcr} x_1 - \frac{5}{12}x_4 & = & -\frac{5}{12} \\ x_2 - \frac{1}{4}x_4 & = & \frac{3}{4} \\ x_3 + \frac{5}{24}x_4 & = & \frac{35}{24} \end{array} \right.$$

We have that x_4 is free and we assign it the parameter t. We obtain $x_3 = -\frac{5}{24}t + \frac{35}{24}$, $x_2 = \frac{1}{4}t + \frac{3}{4}$, and $x_1 = \frac{5}{12}t - \frac{5}{12}$. Our solution is $\left\{\left(\frac{5}{12}t - \frac{5}{12}, \frac{1}{4}t + \frac{3}{4}, -\frac{5}{24}t + \frac{35}{24}, t\right) : -\infty < t < \infty\right\}$ and leave it to the reader to check. □

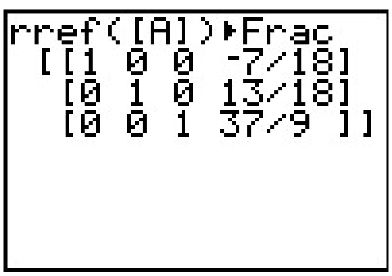
Like all good algorithms, putting a matrix in row echelon or reduced row echelon form can easily be programmed into a calculator, and, doubtless, your graphing calculator has such a feature. We use this in our next example.

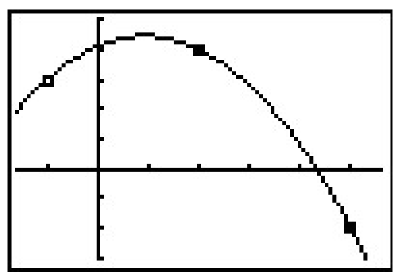
Example 8.2.3. Find the quadratic function passing through the points $(-1,3)$, $(2,4)$, $(5,-2)$.

Solution. According to Definition 2.5, a quadratic function has the form $f(x) = ax^2+bx+c$ where $a \neq 0$. Our goal is to find a, b and c so that the three given points are on the graph of f. If $(-1,3)$ is on the graph of f, then $f(-1) = 3$, or $a(-1)^2 + b(-1) + c = 3$ which reduces to $a - b + c = 3$, an honest-to-goodness linear equation with the variables a, b and c. Since the point $(2,4)$ is also on the graph of f, then $f(2) = 4$ which gives us the equation $4a + 2b + c = 4$. Lastly, the point $(5,-2)$ is on the graph of f gives us $25a+5b+c = -2$. Putting these together, we obtain a system of three linear equations. Encoding this into an augmented matrix produces

$$\left\{ \begin{array}{rcr} a - b + c & = & 3 \\ 4a + 2b + c & = & 4 \\ 25a + 5b + c & = & -2 \end{array} \right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 4 \\ 25 & 5 & 1 & -2 \end{array} \right]$$

Using a calculator,[4] we find $a = -\frac{7}{18}$, $b = \frac{13}{18}$ and $c = \frac{37}{9}$. Hence, the one and only quadratic which fits the bill is $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$. To verify this analytically, we see that $f(-1) = 3$, $f(2) = 4$, and $f(5) = -2$. We can use the calculator to check our solution as well by plotting the three data points and the function f.





The graph of $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$
with the points $(-1,3)$, $(2,4)$ and $(5,-2)$

□

[4]We've tortured you enough already with fractions in this exposition!

8.2.1 Exercises

In Exercises 1 - 6, state whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

1. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

5. $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

In Exercises 7 - 12, the following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

7. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

10. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

11. $\left[\begin{array}{cccc|c} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 13 - 26, solve the following systems of linear equations using the techniques discussed in this section. Compare and contrast these techniques with those you used to solve the systems in the Exercises in Section 8.1.

13. $\left\{\begin{array}{rcr} -5x+y & = & 17 \\ x+y & = & 5 \end{array}\right.$

14. $\left\{\begin{array}{rcr} x+y+z & = & 3 \\ 2x-y+z & = & 0 \\ -3x+5y+7z & = & 7 \end{array}\right.$

15. $\left\{\begin{array}{rcr} 4x-y+z & = & 5 \\ 2y+6z & = & 30 \\ x+z & = & 5 \end{array}\right.$

16. $\left\{\begin{array}{rcr} x-2y+3z & = & 7 \\ -3x+y+2z & = & -5 \\ 2x+2y+z & = & 3 \end{array}\right.$

17. $\left\{\begin{array}{rcr} 3x-2y+z & = & -5 \\ x+3y-z & = & 12 \\ x+y+2z & = & 0 \end{array}\right.$

18. $\left\{\begin{array}{rcr} 2x-y+z & = & -1 \\ 4x+3y+5z & = & 1 \\ 5y+3z & = & 4 \end{array}\right.$

19. $\left\{\begin{array}{rcr} x-y+z & = & -4 \\ -3x+2y+4z & = & -5 \\ x-5y+2z & = & -18 \end{array}\right.$

20. $\left\{\begin{array}{rcr} 2x-4y+z & = & -7 \\ x-2y+2z & = & -2 \\ -x+4y-2z & = & 3 \end{array}\right.$

21. $\left\{\begin{array}{rcl} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array}\right.$

22. $\left\{\begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array}\right.$

23. $\left\{\begin{array}{rcr} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{array}\right.$

24. $\left\{\begin{array}{rcr} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{array}\right.$

25. $\left\{\begin{array}{rcr} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{array}\right.$

26. $\left\{\begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array}\right.$

27. It's time for another meal at our local buffet. This time, 22 diners (5 of whom were children) feasted for \$162.25, before taxes. If the kids buffet is \$4.50, the basic buffet is \$7.50, and the deluxe buffet (with crab legs) is \$9.25, find out how many diners chose the deluxe buffet.

28. Carl wants to make a party mix consisting of almonds (which cost \$7 per pound), cashews (which cost \$5 per pound), and peanuts (which cost \$2 per pound.) If he wants to make a 10 pound mix with a budget of \$35, what are the possible combinations almonds, cashews, and peanuts? (You may find it helpful to review Example 8.1.3 in Section 8.1.)

29. Find the quadratic function passing through the points $(-2, 1)$, $(1, 4)$, $(3, -2)$

30. At 9 PM, the temperature was 60°F; at midnight, the temperature was 50°F; and at 6 AM, the temperature was 70°F . Use the technique in Example 8.2.3 to fit a quadratic function to these data with the temperature, T, measured in degrees Fahrenheit, as the dependent variable, and the number of hours after 9 PM, t, measured in hours, as the independent variable. What was the coldest temperature of the night? When did it occur?

31. The price for admission into the Stitz-Zeager Sasquatch Museum and Research Station is \$15 for adults and \$8 for kids 13 years old and younger. When the Zahlenreich family visits the museum their bill is \$38 and when the Nullsatz family visits their bill is \$39. One day both families went together and took an adult babysitter along to watch the kids and the total admission charge was \$92. Later that summer, the adults from both families went without the kids and the bill was \$45. Is that enough information to determine how many adults and children are in each family? If not, state whether the resulting system is inconsistent or consistent dependent. In the latter case, give at least two plausible solutions.

32. Use the technique in Example 8.2.3 to find the line between the points $(-3, 4)$ and $(6, 1)$. How does your answer compare to the slope-intercept form of the line in Equation 2.3?

33. With the help of your classmates, find at least two different row echelon forms for the matrix

$$\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$$

8.2.2 Answers

1. Reduced row echelon form
2. Neither
3. Row echelon form only
4. Reduced row echelon form
5. Reduced row echelon form
6. Row echelon form only
7. $(-2, 7)$
8. $(-3, 20, 19)$
9. $(-3t + 4, -6t - 6, 2, t)$ for all real numbers t
10. Inconsistent
11. $(8s - t + 7, -4s + 3t + 2, s, t)$ for all real numbers s and t
12. $(-9t - 3, 4t + 20, t)$ for all real numbers t
13. $(-2, 7)$
14. $(1, 2, 0)$
15. $(-t + 5, -3t + 15, t)$ for all real numbers t
16. $(2, -1, 1)$
17. $(1, 3, -2)$
18. Inconsistent
19. $(1, 3, -2)$
20. $\left(-3, \frac{1}{2}, 1\right)$
21. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
22. $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$ for all real numbers t
23. Inconsistent
24. $(4, -3, 1)$
25. $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$ for all real numbers t
26. $(1, 2, 3, 4)$
27. This time, 7 diners chose the deluxe buffet.
28. If t represents the amount (in pounds) of peanuts, then we need $1.5t - 7.5$ pounds of almonds and $17.5 - 2.5t$ pounds of cashews. Since we can't have a negative amount of nuts, $5 \leq t \leq 7$.
29. $f(x) = -\frac{4}{5}x^2 + \frac{1}{5}x + \frac{23}{5}$
30. $T(t) = \frac{20}{27}t^2 - \frac{50}{9}t + 60$. Lowest temperature of the evening $\frac{595}{12} \approx 49.58^{\circ}$F at 12:45 AM.

31. Let x_1 and x_2 be the numbers of adults and children, respectively, in the Zahlenreich family and let x_3 and x_4 be the numbers of adults and children, respectively, in the Nullsatz family. The system of equations determined by the given information is

$$\left\{ \begin{array}{rcl} 15x_1 + 8x_2 & = & 38 \\ 15x_3 + 8x_4 & = & 39 \\ 15x_1 + 8x_2 + 15x_3 + 8x_4 & = & 77 \\ 15x_1 + 15x_3 & = & 45 \end{array} \right.$$

We subtracted the cost of the babysitter in E3 so the constant is 77, not 92. This system is consistent dependent and its solution is $\left(\frac{8}{15}t + \frac{2}{5}, -t + 4, -\frac{8}{15}t + \frac{13}{5}, t\right)$. Our variables represent numbers of adults and children so they must be whole numbers. Running through the values $t = 0, 1, 2, 3, 4$ yields only one solution where all four variables are whole numbers; $t = 3$ gives us $(2, 1, 1, 3)$. Thus there are 2 adults and 1 child in the Zahlenreichs and 1 adult and 3 kids in the Nullsatzs.

8.3 Matrix Arithmetic

In Section 8.2, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily divorced from systems of linear equations. To do so conveniently requires some more notation. When we write $A = [a_{ij}]_{m \times n}$, we mean A is an m by n matrix[1] and a_{ij} is the entry found in the ith row and jth column. Schematically, we have

$$A = \overset{\text{j counts columns from left to right}}{\longrightarrow}\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right]\Big\downarrow \quad \text{i counts rows from top to bottom}$$

With this new notation we can define what it means for two matrices to be equal.

Definition 8.6. Matrix Equality: Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times r}$, we write $A = B$ provided

1. $m = p$ and $n = r$
2. $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots.[2] For example, the two 2×3 matrices below are, despite appearances, equal.

$$\left[\begin{array}{rrr} 0 & -2 & 9 \\ 25 & 117 & -3 \end{array}\right] = \left[\begin{array}{ccc} \ln(1) & \sqrt[3]{-8} & e^{2\ln(3)} \\ 125^{2/3} & 3^2 \cdot 13 & \log(0.001) \end{array}\right]$$

Now that we have an agreed upon understanding of what it means for two matrices to equal each other, we may begin defining arithmetic operations on matrices. Our first operation is addition.

Definition 8.7. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, we define

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$

As an example, consider the sum below.

[1] Recall that means A has m rows and n columns.

[2] Critics may well ask: Why not leave it at that? Why the need for all the notation in Definition 8.6? It is the authors' attempt to expose you to the wonderful world of mathematical precision.

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ -5 & -3 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+4 \\ 4+(-5) & (-1)+(-3) \\ 0+8 & (-7)+1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -1 & -4 \\ 8 & -6 \end{bmatrix}$$

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A + B = B + A$ for matrices A and B, provided they are the same size so that the sum is defined in the first place. This is the **commutative property** of matrix addition. To see why this is true in general, we appeal to the definition of matrix addition. Given $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$,

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n} = [b_{ij} + a_{ij}]_{m\times n} = [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = B + A$$

where the second equality is the definition of $A + B$, the third equality holds by the commutative law of real number addition, and the fourth equality is the definition of $B + A$. In other words, matrix addition is commutative because real number addition is. A similar argument shows the **associative property** of matrix addition also holds, inherited in turn from the associative law of real number addition. Specifically, for matrices A, B, and C of the same size, $(A + B) + C = A + (B + C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means that we can write $A + B + C$ without parentheses and there is no ambiguity as to what this means.[3] These properties and more are summarized in the following theorem.

Theorem 8.3. Properties of Matrix Addition

- **Commutative Property:** For all $m \times n$ matrices, $A + B = B + A$
- **Associative Property:** For all $m \times n$ matrices, $(A + B) + C = A + (B + C)$
- **Identity Property:** If $0_{m\times n}$ is the $m \times n$ matrix whose entries are all 0, then $0_{m\times n}$ is called the ***m* × *n* additive identity** and for all $m \times n$ matrices A

$$A + 0_{m\times n} = 0_{m\times n} + A = A$$

- **Inverse Property:** For every given $m \times n$ matrix A, there is a unique matrix denoted $-A$ called the **additive inverse of *A*** such that

$$A + (-A) = (-A) + A = 0_{m\times n}$$

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices. To establish the inverse property, given a matrix $A = [a_{ij}]_{m\times n}$, we are looking for a matrix $B = [b_{ij}]_{m\times n}$ so that $A + B = 0_{m\times n}$. By the definition of matrix addition, we must

[3] A technical detail which is sadly lost on most readers.

have that $a_{ij} + b_{ij} = 0$ for all i and j. Solving, we get $b_{ij} = -a_{ij}$. Hence, given a matrix A, its additive inverse, which we call $-A$, does exist and is unique and, moreover, is given by the formula: $-A = [-a_{ij}]_{m\times n}$. The long and short of this is: to get the additive inverse of a matrix, take additive inverses of each of its entries. With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a - b = a + (-b)$; that is, subtraction is defined as 'adding the opposite (inverse).' We extend this concept to matrices. For two matrices A and B of the same size, we define $A - B = A + (-B)$. At the level of entries, this amounts to

$$A - B = A + (-B) = [a_{ij}]_{m\times n} + [-b_{ij}]_{m\times n} = [a_{ij} + (-b_{ij})]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$$

Thus to subtract two matrices of equal size, we subtract their corresponding entries. Surprised?

Our next task is to define what it means to multiply a matrix by a real number. Thinking back to arithmetic, you may recall that multiplication, at least by a natural number, can be thought of as 'rapid addition.' For example, $2 + 2 + 2 = 3 \cdot 2$. We know from algebra[4] that $3x = x + x + x$, so it seems natural that given a matrix A, we define $3A = A + A + A$. If $A = [a_{ij}]_{m\times n}$, we have

$$3A = A + A + A = [a_{ij}]_{m\times n} + [a_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = [a_{ij} + a_{ij} + a_{ij}]_{m\times n} = [3a_{ij}]_{m\times n}$$

In other words, multiplying the *matrix* in this fashion by 3 is the same as multiplying *each entry* by 3. This leads us to the following definition.

Definition 8.8. Scalar[a] Multiplication: We define the product of a real number and a matrix to be the matrix obtained by multiplying each of its entries by said real number. More specifically, if k is a real number and $A = [a_{ij}]_{m\times n}$, we define

$$kA = k\,[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$$

[a]The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar).

One may well wonder why the word 'scalar' is used for 'real number.' It has everything to do with 'scaling' factors.[5] A point $P(x, y)$ in the plane can be represented by its position matrix, P:

$$(x, y) \leftrightarrow P = \left[\begin{array}{c} x \\ y \end{array}\right]$$

Suppose we take the point $(-2, 1)$ and multiply its position matrix by 3. We have

$$3P = 3\left[\begin{array}{r} -2 \\ 1 \end{array}\right] = \left[\begin{array}{r} 3(-2) \\ 3(1) \end{array}\right] = \left[\begin{array}{r} -6 \\ 3 \end{array}\right]$$

which corresponds to the point $(-6, 3)$. We can imagine taking $(-2, 1)$ to $(-6, 3)$ in this fashion as a dilation by a factor of 3 in both the horizontal and vertical directions. Doing this to all points (x, y) in the plane, therefore, has the effect of magnifying (scaling) the plane by a factor of 3.

[4]The Distributive Property, in particular.

[5]See Section 1.7.

As did matrix addition, scalar multiplication inherits many properties from real number arithmetic. Below we summarize these properties.

Theorem 8.4. Properties of Scalar Multiplication

- **Associative Property:** For every $m \times n$ matrix A and scalars k and r, $(kr)A = k(rA)$.
- **Identity Property:** For all $m \times n$ matrices A, $1A = A$.
- **Additive Inverse Property:** For all $m \times n$ matrices A, $-A = (-1)A$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every $m \times n$ matrix A and scalars k and r,

$$(k + r)A = kA + rA$$

- **Distributive Property of Scalar Multiplication over Matrix Addition:** For all $m \times n$ matrices A and B scalars k,

$$k(A + B) = kA + kB$$

- **Zero Product Property:** If A is an $m \times n$ matrix and k is a scalar, then

$$kA = 0_{m\times n} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad A = 0_{m\times n}$$

As with the other results in this section, Theorem 8.4 can be proved using the definitions of scalar multiplication and matrix addition. For example, to prove that $k(A + B) = kA + kB$ for a scalar k and $m \times n$ matrices A and B, we start by adding A and B, then multiplying by k and seeing how that compares with the sum of kA and kB.

$$k(A + B) = k\left([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\right) = k\left[a_{ij} + b_{ij}\right]_{m\times n} = \left[k\left(a_{ij} + b_{ij}\right)\right]_{m\times n} = \left[ka_{ij} + kb_{ij}\right]_{m\times n}$$

As for $kA + kB$, we have

$$kA + kB = k\left[a_{ij}\right]_{m\times n} + k\left[b_{ij}\right]_{m\times n} = \left[ka_{ij}\right]_{m\times n} + \left[kb_{ij}\right]_{m\times n} = \left[ka_{ij} + kb_{ij}\right]_{m\times n} \ \checkmark$$

which establishes the property. The remaining properties are left to the reader. The properties in Theorems 8.3 and 8.4 establish an algebraic system that lets us treat matrices and scalars more or less as we would real numbers and variables, as the next example illustrates.

Example 8.3.1. Solve for the matrix A: $3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A\right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix}$ using the definitions and properties of matrix arithmetic.

Solution.

$$
\begin{aligned}
3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix} \\
3A + \left\{ - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) \right\} &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{3}\right)(9) & \left(\frac{1}{3}\right)(12) \\ \left(\frac{1}{3}\right)(-3) & \left(\frac{1}{3}\right)(39) \end{bmatrix} \\
3A + (-1) \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & 13 \end{bmatrix} \\
3A + \left\{ (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) \right\} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} (-1)(2) & (-1)(-1) \\ (-1)(3) & (-1)(5) \end{bmatrix} + ((-1)(5))A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + (-5)A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-5)A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
(3 + (-5))A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) \\
(-2)A + 0_{2\times 2} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \\
(-2)A &= \begin{bmatrix} -1-(-2) & 6-1 \\ 5-(-3) & 11-(-5) \end{bmatrix} \\
(-2)A &= \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(-\frac{1}{2}\right)((-2)A) &= -\frac{1}{2} \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(\left(-\frac{1}{2}\right)(-2)\right) A &= \begin{bmatrix} \left(-\frac{1}{2}\right)(1) & \left(-\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(8) & \left(-\frac{1}{2}\right)(16) \end{bmatrix} \\
1A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -\frac{16}{2} \end{bmatrix} \\
A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -8 \end{bmatrix}
\end{aligned}
$$

The reader is encouraged to check our answer in the original equation. □

While the solution to the previous example is written in excruciating detail, in practice many of the steps above are omitted. We have spelled out each step in this example to encourage the reader to justify each step using the definitions and properties we have established thus far for matrix arithmetic. The reader is encouraged to solve the equation in Example 8.3.1 as they would any other linear equation, for example: $3a - (2 + 5a) = -4 + \frac{1}{3}(9)$.

We now turn our attention to **matrix multiplication** - that is, multiplying a matrix by another matrix. Based on the 'no surprises' trend so far in the section, you may expect that in order to multiply two matrices, they must be of the same size and you find the product by multiplying the corresponding entries. While this kind of product is used in other areas of mathematics,[6] we define matrix multiplication to serve us in solving systems of linear equations. To that end, we begin by defining the product of a row and a column. We motivate the general definition with an example. Consider the two matrices A and B below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

Let $R1$ denote the first row of A and $C1$ denote the first column of B. To find the 'product' of $R1$ with $C1$, denoted $R1 \cdot C1$, we first find the product of the first entry in $R1$ and the first entry in $C1$. Next, we add to that the product of the second entry in $R1$ and the second entry in $C1$. Finally, we take that sum and we add to that the product of the last entry in $R1$ and the last entry in $C1$. Using entry notation, $R1 \cdot C1 = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (2)(3) + (0)(4) + (-1)(5) = 6 + 0 + (-5) = 1$. We can visualize this schematically as follows

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

$$\underbrace{\begin{matrix} \boxed{2} & 0 & -1 \end{matrix} \begin{matrix} \boxed{3} \\ 4 \\ 5 \end{matrix}}_{\substack{a_{11}b_{11} \\ (2)(3)}} \quad \begin{matrix} + \\ + \end{matrix} \quad \underbrace{\begin{matrix} 2 & \boxed{0} & -1 \end{matrix} \begin{matrix} 3 \\ \boxed{4} \\ 5 \end{matrix}}_{\substack{a_{12}b_{21} \\ (0)(4)}} \quad \begin{matrix} + \\ + \end{matrix} \quad \underbrace{\begin{matrix} 2 & 0 & \boxed{-1} \end{matrix} \begin{matrix} 3 \\ 4 \\ \boxed{5} \end{matrix}}_{\substack{a_{13}b_{31} \\ (-1)(5)}}$$

To find $R2 \cdot C3$ where $R2$ denotes the second row of A and $C3$ denotes the third column of B, we proceed similarly. We start with finding the product of the first entry of $R2$ with the first entry in $C3$ then add to it the product of the second entry in $R2$ with the second entry in $C3$, and so forth. Using entry notation, we have $R2 \cdot C3 = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-10)(2) + (3)(-5) + (5)(-2) = -45$. Schematically,

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

[6]See this article on the Hadamard Product.

$$\underbrace{\begin{array}{cccc} \multicolumn{3}{c}{\longrightarrow} & \boxed{2} \\ \boxed{-10} & 3 & 5 & -5 \\ & & & -2 \end{array}\downarrow}_{a_{21}b_{13} = (-10)(2) = -20} + \underbrace{\begin{array}{cccc} \multicolumn{3}{c}{\longrightarrow} & 2 \\ -10 & \boxed{3} & 5 & \boxed{-5} \\ & & & -2 \end{array}\downarrow}_{a_{22}b_{23} = (3)(-5) = -15} + \underbrace{\begin{array}{cccc} \multicolumn{3}{c}{\longrightarrow} & 2 \\ -10 & 3 & \boxed{5} & -5 \\ & & & \boxed{-2} \end{array}\downarrow}_{a_{23}b_{33} = (5)(-2) = -10}$$

Generalizing this process, we have the following definition.

Definition 8.9. Product of a Row and a Column: Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$, denoted $\boldsymbol{R_i \cdot C_j}$** is the real number defined by

$$Ri \cdot Cj = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{in}b_{nj}$$

Note that in order to multiply a row by a column, the number of entries in the row must match the number of entries in the column. We are now in the position to define matrix multiplication.

Definition 8.10. Matrix Multiplication: Suppose $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of $\boldsymbol{A}$ and $\boldsymbol{B}$**, denoted AB, is the matrix defined by

$$AB = [Ri \cdot Cj]_{m\times r}$$

that is

$$AB = \left[\begin{array}{cccc} R1\cdot C1 & R1\cdot C2 & \ldots & R1\cdot Cr \\ R2\cdot C1 & R2\cdot C2 & \ldots & R2\cdot Cr \\ \vdots & \vdots & & \vdots \\ Rm\cdot C1 & Rm\cdot C2 & \ldots & Rm\cdot Cr \end{array}\right]$$

There are a number of subtleties in Definition 8.10 which warrant closer inspection. First and foremost, Definition 8.10 tells us that the ij-entry of a matrix product AB is the ith row of A times the jth column of B. In order for this to be defined, the number of entries in the rows of A must match the number of entries in the columns of B. This means that the number of columns of A must match[7] the number of rows of B. In other words, to multiply A times B, the second dimension of A must match the first dimension of B, which is why in Definition 8.10, $A_{m\times \underline{n}}$ is being multiplied by a matrix $B_{\underline{n}\times r}$. Furthermore, the product matrix AB has as many rows as A and as many columns of B. As a result, when multiplying a matrix $A_{\underline{m}\times n}$ by a matrix $B_{n\times \underline{r}}$, the result is the matrix $AB_{\underline{m}\times \underline{r}}$. Returning to our example matrices below, we see that A is a $2\times \underline{3}$ matrix and B is a $\underline{3}\times 4$ matrix. This means that the product matrix AB is defined and will be a 2×4 matrix.

$$A = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array}\right] \quad B = \left[\begin{array}{rrrr} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{array}\right]$$

[7]The reader is encouraged to think this through carefully.

Using Ri to denote the ith row of A and Cj to denote the jth column of B, we form AB according to Definition 8.10.

$$AB = \begin{bmatrix} R1 \cdot C1 & R1 \cdot C2 & R1 \cdot C3 & R1 \cdot C4 \\ R2 \cdot C1 & R2 \cdot C2 & R2 \cdot C3 & R2 \cdot C4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 & -4 \\ 7 & 14 & -45 & 47 \end{bmatrix}$$

Note that the product BA is not defined, since B is a $3 \times \underline{4}$ matrix while A is a $\underline{2} \times 3$ matrix; B has more columns than A has rows, and so it is not possible to multiply a row of B by a column of A. Even when the dimensions of A and B are compatible such that AB and BA are both defined, the product AB and BA aren't necessarily equal.[8] In other words, AB may not equal BA. Although there is no commutative property of matrix multiplication in general, several other real number properties are inherited by matrix multiplication, as illustrated in our next theorem.

Theorem 8.5. Properties of Matrix Multiplication Let A, B and C be matrices such that all of the matrix products below are defined and let k be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$
- **Associative Property with Scalar Multiplication:** $k(AB) = (kA)B = A(kB)$
- **Identity Property:** For a natural number k, the $\boldsymbol{k \times k}$ **identity matrix**, denoted I_k, is defined by $I_k = [d_{ij}]_{k \times k}$ where

$$d_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

  For all $m \times n$ matrices, $I_m A = A I_n = A$.

- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$

The one property in Theorem 8.5 which begs further investigation is, without doubt, the multiplicative identity. The entries in a matrix where $i = j$ comprise what is called the **main diagonal** of the matrix. The identity matrix has 1's along its main diagonal and 0's everywhere else. A few examples of the matrix I_k mentioned in Theorem 8.5 are given below. The reader is encouraged to see how they match the definition of the identity matrix presented there.

$$\underset{I_1}{[1]} \quad \underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad \underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \underset{I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

[8] And may not even have the same dimensions. For example, if A is a 2×3 matrix and B is a 3×2 matrix, then AB is defined and is a 2×2 matrix while BA is also defined... but is a 3×3 matrix!

The identity matrix is an example of what is called a **square matrix** as it has the same number of rows as columns. Note that to in order to verify that the identity matrix acts as a multiplicative identity, some care must be taken depending on the order of the multiplication. For example, take the matrix 2×3 matrix A from earlier

$$A = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

In order for the product $I_k A$ to be defined, $k = 2$; similarly, for AI_k to be defined, $k = 3$. We leave it to the reader to show $I_2 A = A$ and $AI_3 = A$. In other words,

$$\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

and

$$\left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

While the proofs of the properties in Theorem 8.5 are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following example provides some practice with matrix multiplication and its properties. As usual, some valuable lessons are to be learned.

Example 8.3.2.

1. Find AB for $A = \left[\begin{array}{rrr} -23 & -1 & 17 \\ 46 & 2 & -34 \end{array} \right]$ and $B = \left[\begin{array}{rr} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{array} \right]$

2. Find $C^2 - 5C + 10I_2$ for $C = \left[\begin{array}{rr} 1 & -2 \\ 3 & 4 \end{array} \right]$

3. Suppose M is a 4×4 matrix. Use Theorem 8.5 to expand $(M - 2I_4)(M + 3I_4)$.

Solution.

1. We have $AB = \left[\begin{array}{rrr} -23 & -1 & 17 \\ 46 & 2 & -34 \end{array} \right] \left[\begin{array}{rr} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{array} \right] = \left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \end{array} \right]$

2. Just as x^2 means x times itself, C^2 denotes the matrix C times itself. We get

$$\begin{aligned}
C^2 - 5C + 10I_2 &= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

3. We expand $(M - 2I_4)(M + 3I_4)$ with the same pedantic zeal we showed in Example 8.3.1. The reader is encouraged to determine which property of matrix arithmetic is used as we proceed from one step to the next.

$$\begin{aligned}
(M - 2I_4)(M + 3I_4) &= (M - 2I_4)M + (M - 2I_4)(3I_4) \\
&= MM - (2I_4)M + M(3I_4) - (2I_4)(3I_4) \\
&= M^2 - 2(I_4M) + 3(MI_4) - 2(I_4(3I_4)) \\
&= M^2 - 2M + 3M - 2(3(I_4I_4)) \\
&= M^2 + M - 6I_4
\end{aligned}$$

□

Example 8.3.2 illustrates some interesting features of matrix multiplication. First note that in part 1, neither A nor B is the zero matrix, yet the product AB is the zero matrix. Hence, the the zero product property enjoyed by real numbers and scalar multiplication does not hold for matrix multiplication. Parts 2 and 3 introduce us to polynomials involving matrices. The reader is encouraged to step back and compare our expansion of the matrix product $(M - 2I_4)(M + 3I_4)$ in part 3 with the product $(x - 2)(x + 3)$ from real number algebra. The exercises explore this kind of parallel further.

As we mentioned earlier, a point $P(x, y)$ in the xy-plane can be represented as a 2×1 position matrix. We now show that matrix multiplication can be used to rotate these points, and hence graphs of equations.

Example 8.3.3. Let $R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

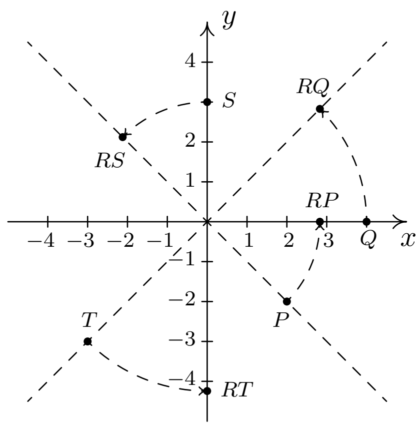
1. Plot $P(2, -2)$, $Q(4, 0)$, $S(0, 3)$, and $T(-3, -3)$ in the plane as well as the points RP, RQ, RS, and RT. Plot the lines $y = x$ and $y = -x$ as guides. What does R appear to be doing to these points?

2. If a point P is on the hyperbola $x^2 - y^2 = 4$, show that the point RP is on the curve $y = \frac{2}{x}$.

Solution. For $P(2,-2)$, the position matrix is $P = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, and

$$\begin{aligned} RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \end{aligned}$$

We have that R takes $(2,-2)$ to $(2\sqrt{2},0)$. Similarly, we find $(4,0)$ is moved to $(2\sqrt{2},2\sqrt{2})$, $(0,3)$ is moved to $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, and $(-3,-3)$ is moved to $(0,-3\sqrt{2})$. Plotting these in the coordinate plane along with the lines $y=x$ and $y=-x$, we see that the matrix R is rotating these points counterclockwise by 45°.



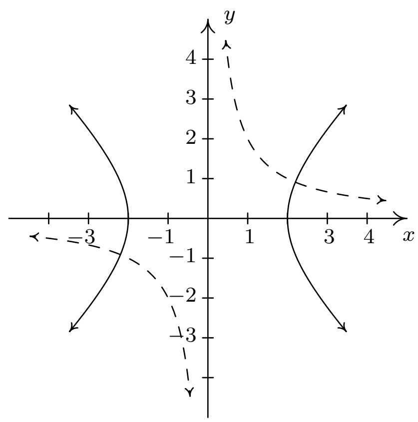
For a generic point $P(x,y)$ on the hyperbola $x^2 - y^2 = 4$, we have

$$\begin{aligned} RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix} \end{aligned}$$

which means R takes (x,y) to $\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)$. To show that this point is on the curve $y = \frac{2}{x}$, we replace x with $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and y with $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ and simplify.

$$\begin{aligned} y &= \tfrac{2}{x} \\ \tfrac{\sqrt{2}}{2}x + \tfrac{\sqrt{2}}{2}y &\stackrel{?}{=} \tfrac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y} \\ \left(\tfrac{\sqrt{2}}{2}x - \tfrac{\sqrt{2}}{2}y\right)\left(\tfrac{\sqrt{2}}{2}x + \tfrac{\sqrt{2}}{2}y\right) &\stackrel{?}{=} \left(\frac{2}{\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y}\right)\left(\tfrac{\sqrt{2}}{2}x - \tfrac{\sqrt{2}}{2}y\right) \\ \left(\tfrac{\sqrt{2}}{2}x\right)^2 - \left(\tfrac{\sqrt{2}}{2}y\right)^2 &\stackrel{?}{=} 2 \\ \tfrac{x^2}{2} - \tfrac{y^2}{2} &\stackrel{?}{=} 2 \\ x^2 - y^2 &\stackrel{\checkmark}{=} 4 \end{aligned}$$

Since (x, y) is on the hyperbola $x^2 - y^2 = 4$, we know that this last equation is true. Since all of our steps are reversible, this last equation is equivalent to our original equation, which establishes the point is, indeed, on the graph of $y = \frac{2}{x}$. This means the graph of $y = \frac{2}{x}$ is a hyperbola, and it is none other than the hyperbola $x^2 - y^2 = 4$ rotated counterclockwise by $45°$.[9] Below we have the graph of $x^2 - y^2 = 4$ (solid line) and $y = \frac{2}{x}$ (dashed line) for comparison.



□

When we started this section, we mentioned that we would temporarily consider matrices as their own entities, but that the algebra developed here would ultimately allow us to solve systems of linear equations. To that end, consider the system

$$\begin{cases} 3x - y + z = 8 \\ x + 2y - z = 4 \\ 2x + 3y - 4z = 10 \end{cases}$$

In Section 8.2, we encoded this system into the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

[9]See Section 7.5 for more details.

Recall that the entries to the left of the vertical line come from the coefficients of the variables in the system, while those on the right comprise the associated constants. For that reason, we may form the **coefficient matrix** A, the **unknowns matrix** X and the **constant matrix** B as below

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}$$

We now consider the matrix equation $AX = B$.

$$\begin{aligned} AX &= B \\ \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 3x - y + z \\ x + 2y - z \\ 2x + 3y - 4z \end{bmatrix} &= \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix} \end{aligned}$$

We see that finding a solution (x, y, z) to the original system corresponds to finding a solution X for the matrix equation $AX = B$. If we think about solving the real number equation $ax = b$, we would simply 'divide' both sides by a. Is it possible to 'divide' both sides of the matrix equation $AX = B$ by the matrix A? This is the central topic of Section 8.4.

8.3.1 Exercises

For each pair of matrices A and B in Exercises 1 - 7, find the following, if defined

- $3A$
- $-B$
- A^2
- $A-2B$
- AB
- BA

1. $A=\begin{bmatrix}2 & -3\\ 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & -2\\ 4 & 8\end{bmatrix}$

2. $A=\begin{bmatrix}-1 & 5\\ -3 & 6\end{bmatrix}$, $B=\begin{bmatrix}2 & 10\\ -7 & 1\end{bmatrix}$

3. $A=\begin{bmatrix}-1 & 3\\ 5 & 2\end{bmatrix}$, $B=\begin{bmatrix}7 & 0 & 8\\ -3 & 1 & 4\end{bmatrix}$

4. $A=\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}$, $B=\begin{bmatrix}-1 & 3 & -5\\ 7 & -9 & 11\end{bmatrix}$

5. $A=\begin{bmatrix}7\\ 8\\ 9\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$

6. $A=\begin{bmatrix}1 & -2\\ -3 & 4\\ 5 & -6\end{bmatrix}$, $B=\begin{bmatrix}-5 & 1 & 8\end{bmatrix}$

7. $A=\begin{bmatrix}2 & -3 & 5\\ 3 & 1 & -2\\ -7 & 1 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 1\\ 17 & 33 & 19\\ 10 & 19 & 11\end{bmatrix}$

In Exercises 8 - 21, use the matrices

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \quad B=\begin{bmatrix}0 & -3\\ -5 & 2\end{bmatrix} \quad C=\begin{bmatrix}10 & -\frac{11}{2} & 0\\ \frac{3}{5} & 5 & 9\end{bmatrix}$$

$$D=\begin{bmatrix}7 & -13\\ -\frac{4}{3} & 0\\ 6 & 8\end{bmatrix} \quad E=\begin{bmatrix}1 & 2 & 3\\ 0 & 4 & -9\\ 0 & 0 & -5\end{bmatrix}$$

to compute the following or state that the indicated operation is undefined.

8. $7B-4A$
9. AB
10. BA
11. $E+D$
12. ED
13. $CD+2I_2A$
14. $A-4I_2$
15. A^2-B^2
16. $(A+B)(A-B)$
17. $A^2-5A-2I_2$
18. $E^2+5E-36I_3$
19. EDC
20. CDE
21. $ABCEDI_2$
22. Let $A=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$ $E_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $E_2=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}$ $E_3=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$

    Compute E_1A, E_2A and E_3A. What effect did each of the E_i matrices have on the rows of A? Create E_4 so that its effect on A is to multiply the bottom row by -6. How would you extend this idea to matrices with more than two rows?

In Exercises 23 - 29, consider the following scenario. In the small village of Pedimaxus in the country of Sasquatchia, all 150 residents get one of the two local newspapers. Market research has shown that in any given week, 90% of those who subscribe to the Pedimaxus Tribune want to keep getting it, but 10% want to switch to the Sasquatchia Picayune. Of those who receive the Picayune, 80% want to continue with it and 20% want switch to the Tribune. We can express this situation using matrices. Specifically, let X be the 'state matrix' given by

$$X = \begin{bmatrix} T \\ P \end{bmatrix}$$

where T is the number of people who get the Tribune and P is the number of people who get the Picayune in a given week. Let Q be the 'transition matrix' given by

$$Q = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

such that QX will be the state matrix for the next week.

23. Let's assume that when Pedimaxus was founded, all 150 residents got the Tribune. (Let's call this Week 0.) This would mean

$$X = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$$

Since 10% of that 150 want to switch to the Picayune, we should have that for Week 1, 135 people get the Tribune and 15 people get the Picayune. Show that QX in this situation is indeed

$$QX = \begin{bmatrix} 135 \\ 15 \end{bmatrix}$$

24. Assuming that the percentages stay the same, we can get to the subscription numbers for Week 2 by computing Q^2X. How many people get each paper in Week 2?

25. Explain why the transition matrix does what we want it to do.

26. If the conditions do not change from week to week, then Q remains the same and we have what's known as a **Stochastic Process**[10] because Week n's numbers are found by computing Q^nX. Choose a few values of n and, with the help of your classmates and calculator, find out how many people get each paper for that week. You should start to see a pattern as $n \to \infty$.

27. If you didn't see the pattern, we'll help you out. Let

$$X_s = \begin{bmatrix} 100 \\ 50 \end{bmatrix}.$$

Show that $QX_s = X_s$ This is called the **steady state** because the number of people who get each paper didn't change for the next week. Show that $Q^nX \to X_s$ as $n \to \infty$.

[10]More specifically, we have a Markov Chain, which is a special type of stochastic process.

28. Now let

$$S = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Show that $Q^n \to S$ as $n \to \infty$.

29. Show that $SY = X_s$ for any matrix Y of the form

$$Y = \begin{bmatrix} y \\ 150 - y \end{bmatrix}$$

This means that no matter how the distribution starts in Pedimaxus, if Q is applied often enough, we always end up with 100 people getting the Tribune and 50 people getting the Picayune.

30. Let $z = a + bi$ and $w = c + di$ be arbitrary complex numbers. Associate z and w with the matrices

$$Z = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \text{ and } W = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Show that complex number addition, subtraction and multiplication are mirrored by the associated *matrix* arithmetic. That is, show that $Z + W$, $Z - W$ and ZW produce matrices which can be associated with the complex numbers $z + w$, $z - w$ and zw, respectively.

31. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -3 \\ -5 & 2 \end{bmatrix}$$

Compare $(A + B)^2$ to $A^2 + 2AB + B^2$. Discuss with your classmates what constraints must be placed on two arbitrary matrices A and B so that both $(A+B)^2$ and $A^2+2AB+B^2$ exist. When will $(A+B)^2 = A^2+2AB+B^2$? In general, what is the correct formula for $(A+B)^2$?

In Exercises 32 - 36, consider the following definitions. A square matrix is said to be an **upper triangular matrix** if all of its entries below the main diagonal are zero and it is said to be a **lower triangular matrix** if all of its entries above the main diagonal are zero. For example,

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -9 \\ 0 & 0 & -5 \end{bmatrix}$$

from Exercises 8 - 21 above is an upper triangular matrix whereas

$$F = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

is a lower triangular matrix. (Zeros are allowed on the main diagonal.) Discuss the following questions with your classmates.

32. Give an example of a matrix which is neither upper triangular nor lower triangular.

33. Is the product of two $n \times n$ upper triangular matrices always upper triangular?

34. Is the product of two $n \times n$ lower triangular matrices always lower triangular?

35. Given the matrix
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
write A as LU where L is a lower triangular matrix and U is an upper triangular matrix?

36. Are there any matrices which are simultaneously upper and lower triangular?

8.3.2 Answers

1. For $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 4 & 8 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & -9 \\ 3 & 12 \end{bmatrix}$
- $-B = \begin{bmatrix} -5 & 2 \\ -4 & -8 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 1 & -18 \\ 6 & 13 \end{bmatrix}$
- $A - 2B = \begin{bmatrix} -8 & 1 \\ -7 & -12 \end{bmatrix}$
- $AB = \begin{bmatrix} -2 & -28 \\ 21 & 30 \end{bmatrix}$
- $BA = \begin{bmatrix} 8 & -23 \\ 16 & 20 \end{bmatrix}$

2. For $A = \begin{bmatrix} -1 & 5 \\ -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 10 \\ -7 & 1 \end{bmatrix}$

- $3A = \begin{bmatrix} -3 & 15 \\ -9 & 18 \end{bmatrix}$
- $-B = \begin{bmatrix} -2 & -10 \\ 7 & -1 \end{bmatrix}$
- $A^2 = \begin{bmatrix} -14 & 25 \\ -15 & 21 \end{bmatrix}$
- $A - 2B = \begin{bmatrix} -5 & -15 \\ 11 & 4 \end{bmatrix}$
- $AB = \begin{bmatrix} -37 & -5 \\ -48 & -24 \end{bmatrix}$
- $BA = \begin{bmatrix} -32 & 70 \\ 4 & -29 \end{bmatrix}$

3. For $A = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 & 8 \\ -3 & 1 & 4 \end{bmatrix}$

- $3A = \begin{bmatrix} -3 & 9 \\ 15 & 6 \end{bmatrix}$
- $-B = \begin{bmatrix} -7 & 0 & -8 \\ 3 & -1 & -4 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 16 & 3 \\ 5 & 19 \end{bmatrix}$
- $A - 2B$ is not defined
- $AB = \begin{bmatrix} -16 & 3 & 4 \\ 29 & 2 & 48 \end{bmatrix}$
- BA is not defined

4. For $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & -5 \\ 7 & -9 & 11 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & 12 \\ 18 & 24 \end{bmatrix}$
- $-B = \begin{bmatrix} 1 & -3 & 5 \\ -7 & 9 & -11 \end{bmatrix}$
- $A^2 = \begin{bmatrix} 28 & 40 \\ 60 & 88 \end{bmatrix}$
- $A - 2B$ is not defined
- $AB = \begin{bmatrix} 26 & -30 & 34 \\ 50 & -54 & 58 \end{bmatrix}$
- BA is not defined

5. For $A = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

- $3A = \begin{bmatrix} 21 \\ 24 \\ 27 \end{bmatrix}$
- $-B = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$
- A^2 is not defined
- $A - 2B$ is not defined
- $AB = \begin{bmatrix} 7 & 14 & 21 \\ 8 & 16 & 24 \\ 9 & 18 & 27 \end{bmatrix}$
- $BA = [50]$

6. For $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 1 & 8 \end{bmatrix}$

- $3A = \begin{bmatrix} 3 & -6 \\ -9 & 12 \\ 15 & -18 \end{bmatrix}$
- $-B = \begin{bmatrix} 5 & -1 & -8 \end{bmatrix}$
- A^2 is not defined
- $A - 2B$ is not defined
- AB is not defined
- $BA = \begin{bmatrix} 32 & -34 \end{bmatrix}$

7. For $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

- $3A = \begin{bmatrix} 6 & -9 & 15 \\ 9 & 3 & -6 \\ -21 & 3 & -3 \end{bmatrix}$
- $-B = \begin{bmatrix} -1 & -2 & -1 \\ -17 & -33 & -19 \\ -10 & -19 & -11 \end{bmatrix}$
- $A^2 = \begin{bmatrix} -40 & -4 & 11 \\ 23 & -10 & 15 \\ -4 & 21 & -36 \end{bmatrix}$
- $A - 2B = \begin{bmatrix} 0 & -7 & 3 \\ -31 & -65 & -40 \\ -27 & -37 & -23 \end{bmatrix}$
- $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $7B - 4A = \begin{bmatrix} -4 & -29 \\ -47 & -2 \end{bmatrix}$

9. $AB = \begin{bmatrix} -10 & 1 \\ -20 & -1 \end{bmatrix}$

10. $BA = \begin{bmatrix} -9 & -12 \\ 1 & -2 \end{bmatrix}$

11. $E + D$ is undefined

12. $ED = \begin{bmatrix} \frac{67}{3} & 11 \\ -\frac{178}{3} & -72 \\ -30 & -40 \end{bmatrix}$

13. $CD + 2I_2A = \begin{bmatrix} \frac{238}{3} & -126 \\ \frac{863}{15} & \frac{361}{5} \end{bmatrix}$

14. $A - 4I_2 = \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix}$

15. $A^2 - B^2 = \begin{bmatrix} -8 & 16 \\ 25 & 3 \end{bmatrix}$

16. $(A+B)(A-B) = \begin{bmatrix} -7 & 3 \\ 46 & 2 \end{bmatrix}$

17. $A^2 - 5A - 2I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. $E^2 + 5E - 36I_3 = \begin{bmatrix} -30 & 20 & -15 \\ 0 & 0 & -36 \\ 0 & 0 & -36 \end{bmatrix}$

19. $EDC = \begin{bmatrix} \frac{3449}{15} & -\frac{407}{6} & 99 \\ -\frac{9548}{15} & -\frac{101}{3} & -648 \\ -324 & -35 & -360 \end{bmatrix}$

20. CDE is undefined

21. $ABCEDI_2 = \begin{bmatrix} -\frac{90749}{15} & -\frac{28867}{5} \\ -\frac{156601}{15} & -\frac{47033}{5} \end{bmatrix}$

22. $E_1A = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$ E_1 interchanged $R1$ and $R2$ of A.

$E_2A = \begin{bmatrix} 5a & 5b & 5c \\ d & e & f \end{bmatrix}$ E_2 multiplied $R1$ of A by 5.

$E_3A = \begin{bmatrix} a-2d & b-2e & c-2f \\ d & e & f \end{bmatrix}$ E_3 replaced $R1$ in A with $R1 - 2R2$.

$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix}$

8.4 Systems of Linear Equations: Matrix Inverses

We concluded Section 8.3 by showing how we can rewrite a system of linear equations as the matrix equation $AX = B$ where A and B are known matrices and the solution matrix X of the equation corresponds to the solution of the system. In this section, we develop the method for solving such an equation. To that end, consider the system

$$\left\{ \begin{array}{rcr} 2x - 3y & = & 16 \\ 3x + 4y & = & 7 \end{array} \right.$$

To write this as a matrix equation, we follow the procedure outlined on page 590. We find the coefficient matrix A, the unknowns matrix X and constant matrix B to be

$$A = \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] \quad X = \left[\begin{array}{c} x \\ y \end{array} \right] \quad B = \left[\begin{array}{r} 16 \\ 7 \end{array} \right]$$

In order to motivate how we solve a matrix equation like $AX = B$, we revisit solving a similar equation involving real numbers. Consider the equation $3x = 5$. To solve, we simply divide both sides by 3 and obtain $x = \frac{5}{3}$. How can we go about defining an analogous process for matrices? To answer this question, we solve $3x = 5$ again, but this time, we pay attention to the properties of real numbers being used at each step. Recall that dividing by 3 is the same as multiplying by $\frac{1}{3} = 3^{-1}$, the so-called *multiplicative inverse*[1] of 3.

$$\begin{array}{rclr} 3x & = & 5 & \\ 3^{-1}(3x) & = & 3^{-1}(5) & \text{Multiply by the (multiplicative) inverse of 3} \\ \left(3^{-1} \cdot 3\right) x & = & 3^{-1}(5) & \text{Associative property of multiplication} \\ 1 \cdot x & = & 3^{-1}(5) & \text{Inverse property} \\ x & = & 3^{-1}(5) & \text{Multiplicative Identity} \end{array}$$

If we wish to check our answer, we substitute $x = 3^{-1}(5)$ into the original equation

$$\begin{array}{rclr} 3x & \stackrel{?}{=} & 5 & \\ 3\left(3^{-1}(5)\right) & \stackrel{?}{=} & 5 & \\ \left(3 \cdot 3^{-1}\right)(5) & \stackrel{?}{=} & 5 & \text{Associative property of multiplication} \\ 1 \cdot 5 & \stackrel{?}{=} & 5 & \text{Inverse property} \\ 5 & \stackrel{\checkmark}{=} & 5 & \text{Multiplicative Identity} \end{array}$$

Thinking back to Theorem 8.5, we know that matrix multiplication enjoys both an associative property and a multiplicative identity. What's missing from the mix is a multiplicative inverse for the coefficient matrix A. Assuming we can find such a beast, we can mimic our solution (and check) to $3x = 5$ as follows

[1]Every nonzero real number a has a multiplicative inverse, denoted a^{-1}, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$.

Solving $AX = B$

$$\begin{array}{rcl} AX & = & B \\ A^{-1}(AX) & = & A^{-1}B \\ \left(A^{-1}A\right)X & = & A^{-1}B \\ I_2X & = & A^{-1}B \\ X & = & A^{-1}B \end{array}$$

Checking our answer

$$\begin{array}{rcl} AX & \stackrel{?}{=} & B \\ A\left(A^{-1}B\right) & \stackrel{?}{=} & B \\ \left(AA^{-1}\right)B & \stackrel{?}{=} & B \\ I_2B & \stackrel{?}{=} & B \\ B & \stackrel{\checkmark}{=} & B \end{array}$$

The matrix A^{-1} is read 'A-inverse' and we will define it formally later in the section. At this stage, we have no idea if such a matrix A^{-1} exists, but that won't deter us from trying to find it.[2] We want A^{-1} to satisfy two equations, $A^{-1}A = I_2$ and $AA^{-1} = I_2$, making A^{-1} necessarily a 2×2 matrix.[3] Hence, we assume A^{-1} has the form

$$A^{-1} = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right]$$

for real numbers x_1, x_2, x_3 and x_4. For reasons which will become clear later, we focus our attention on the equation $AA^{-1} = I_2$. We have

$$\begin{array}{rcl} AA^{-1} & = & I_2 \\ \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array}\right]\left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array}\right] & = & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \\ \left[\begin{array}{cc} 2x_1 - 3x_3 & 2x_2 - 3x_4 \\ 3x_1 + 4x_3 & 3x_2 + 4x_4 \end{array}\right] & = & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \end{array}$$

This gives rise to *two* more systems of equations

$$\left\{\begin{array}{rcl} 2x_1 - 3x_3 & = & 1 \\ 3x_1 + 4x_3 & = & 0 \end{array}\right. \qquad \left\{\begin{array}{rcl} 2x_2 - 3x_4 & = & 0 \\ 3x_2 + 4x_4 & = & 1 \end{array}\right.$$

At this point, it may seem absurd to continue with this venture. After all, the intent was to solve *one* system of equations, and in doing so, we have produced *two* more to solve. Remember, the objective of this discussion is to develop a general *method* which, when used in the correct scenarios, allows us to do far more than just solve a system of equations. If we set about to solve these systems using augmented matrices using the techniques in Section 8.2, we see that not only do both systems have the same coefficient matrix, this coefficient matrix is none other than the matrix A itself. (We will come back to this observation in a moment.)

[2] Much like Carl's quest to find Sasquatch.

[3] Since matrix multiplication isn't necessarily commutative, at this stage, these are two different equations.

$$\left\{ \begin{array}{rcl} 2x_1 - 3x_3 & = & 1 \\ 3x_1 + 4x_3 & = & 0 \end{array} \right. \xrightarrow{\text{Encode into a matrix}} \left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right]$$

$$\left\{ \begin{array}{rcl} 2x_2 - 3x_4 & = & 0 \\ 3x_2 + 4x_4 & = & 1 \end{array} \right. \xrightarrow{\text{Encode into a matrix}} \left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right]$$

To solve these two systems, we use Gauss-Jordan Elimination to put the augmented matrices into reduced row echelon form. (We leave the details to the reader.) For the first system, we get

$$\left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array} \right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{rr|r} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array} \right]$$

which gives $x_1 = \frac{4}{17}$ and $x_3 = -\frac{3}{17}$. To solve the second system, we use the exact same row operations, in the same order, to put its augmented matrix into reduced row echelon form (Think about why that works.) and we obtain

$$\left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array} \right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{rr|r} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array} \right]$$

which means $x_2 = \frac{3}{17}$ and $x_4 = \frac{2}{17}$. Hence,

$$A^{-1} = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right] = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right]$$

We can check to see that A^{-1} behaves as it should by computing AA^{-1}

$$AA^{-1} = \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

As an added bonus,

$$A^{-1}A = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[\begin{array}{rr} 2 & -3 \\ 3 & 4 \end{array} \right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] = I_2 \checkmark$$

We can now return to the problem at hand. From our discussion at the beginning of the section on page 599, we know

$$X = A^{-1}B = \left[\begin{array}{rr} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{array} \right] \left[\begin{array}{r} 16 \\ 7 \end{array} \right] = \left[\begin{array}{r} 5 \\ -2 \end{array} \right]$$

so that our final solution to the system is $(x, y) = (5, -2)$.

As we mentioned, the point of this exercise was not just to solve the system of linear equations, but to develop a general method for finding A^{-1}. We now take a step back and analyze the foregoing discussion in a more general context. In solving for A^{-1}, we used two augmented matrices, both of which contained the same entries as A

$$\left[\begin{array}{rr|r} 2 & -3 & 1 \\ 3 & 4 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & \begin{array}{c} 1 \\ 0 \end{array} \end{array}\right]$$

$$\left[\begin{array}{rr|r} 2 & -3 & 0 \\ 3 & 4 & 1 \end{array}\right] = \left[\begin{array}{c|c} A & \begin{array}{c} 0 \\ 1 \end{array} \end{array}\right]$$

We also note that the reduced row echelon forms of these augmented matrices can be written as

$$\left[\begin{array}{cc|r} 1 & 0 & \frac{4}{17} \\ 0 & 1 & -\frac{3}{17} \end{array}\right] = \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_1 \\ x_3 \end{array} \end{array}\right]$$

$$\left[\begin{array}{cc|r} 1 & 0 & \frac{3}{17} \\ 0 & 1 & \frac{2}{17} \end{array}\right] = \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_2 \\ x_4 \end{array} \end{array}\right]$$

where we have identified the entries to the left of the vertical bar as the identity I_2 and the entries to the right of the vertical bar as the solutions to our systems. The long and short of the solution process can be summarized as

$$\left[\begin{array}{c|c} A & \begin{array}{c} 1 \\ 0 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_1 \\ x_3 \end{array} \end{array}\right]$$

$$\left[\begin{array}{c|c} A & \begin{array}{c} 0 \\ 1 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{c} x_2 \\ x_4 \end{array} \end{array}\right]$$

Since the row operations for both processes are the same, all of the arithmetic on the left hand side of the vertical bar is identical in both problems. The only difference between the two processes is what happens to the constants to the right of the vertical bar. As long as we keep these separated into columns, we can combine our efforts into one 'super-sized' augmented matrix and describe the above process as

$$\left[\begin{array}{c|cc} A & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_2 & \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \end{array}\right]$$

We have the identity matrix I_2 appearing as the right hand side of the first super-sized augmented matrix and the left hand side of the second super-sized augmented matrix. To our surprise and delight, the elements on the right hand side of the second super-sized augmented matrix are none other than those which comprise A^{-1}. Hence, we have

$$\left[\, A \mid I_2 \,\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\, I_2 \mid A^{-1} \,\right]$$

In other words, the process of finding A^{-1} for a matrix A can be viewed as performing a series of row operations which transform A into the identity matrix of the same dimension. We can view this process as follows. In trying to find A^{-1}, we are trying to 'undo' multiplication by the matrix A. The identity matrix in the super-sized augmented matrix $[A|I]$ keeps a running memory of all of the moves required to 'undo' A. This results in exactly what we want, A^{-1}. We are now ready to formalize and generalize the foregoing discussion. We begin with the formal definition of an invertible matrix.

Definition 8.11. An $n \times n$ matrix A is said to be **invertible** if there exists a matrix A^{-1}, read 'A inverse', such that $A^{-1}A = AA^{-1} = I_n$.

Note that, as a consequence of our definition, invertible matrices are square, and as such, the conditions in Definition 8.11 force the matrix A^{-1} to be same dimensions as A, that is, $n \times n$. Since not all matrices are square, not all matrices are invertible. However, just because a matrix is square doesn't guarantee it is invertible. (See the exercises.) Our first result summarizes some of the important characteristics of invertible matrices and their inverses.

Theorem 8.6. Suppose A is an $n \times n$ matrix.

1. If A is invertible then A^{-1} is unique.
2. A is invertible if and only if $AX = B$ has a unique solution for every $n \times r$ matrix B.

The proofs of the properties in Theorem 8.6 rely on a healthy mix of definition and matrix arithmetic. To establish the first property, we assume that A is invertible and suppose the matrices B and C act as inverses for A. That is, $BA = AB = I_n$ and $CA = AC = I_n$. We need to show that B and C are, in fact, the same matrix. To see this, we note that $B = I_nB = (CA)B = C(AB) = CI_n = C$. Hence, any two matrices that act like A^{-1} are, in fact, the same matrix.[4] To prove the second property of Theorem 8.6, we note that if A is invertible then the discussion on page 599 shows the solution to $AX = B$ to be $X = A^{-1}B$, and since A^{-1} is unique, so is $A^{-1}B$. Conversely, if $AX = B$ has a unique solution for every $n \times r$ matrix B, then, in particular, there is a unique solution X_0 to the equation $AX = I_n$. The solution matrix X_0 is our candidate for A^{-1}. We have $AX_0 = I_n$ by definition, but we need to also show $X_0A = I_n$. To that end, we note that $A(X_0A) = (AX_0)A = I_nA = A$. In other words, the matrix X_0A is a solution to the equation $AX = A$. Clearly, $X = I_n$ is also a solution to the equation $AX = A$, and since we are assuming every such equation as a *unique* solution, we must have $X_0A = I_n$. Hence, we have $X_0A = AX_0 = I_n$, so that $X_0 = A^{-1}$ and A is invertible. The foregoing discussion justifies our quest to find A^{-1} using our super-sized augmented matrix approach

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

We are, in essence, trying to find the unique solution to the equation $AX = I_n$ using row operations.

What does all of this mean for a system of linear equations? Theorem 8.6 tells us that if we write the system in the form $AX = B$, then if the coefficient matrix A is invertible, there is only one solution to the system $-$ that is, if A is invertible, the system is consistent and independent.[5] We also know that the process by which we find A^{-1} is determined completely by A, and not by the

[4]If this proof sounds familiar, it should. See the discussion following Theorem 5.2 on page 380.

[5]It can be shown that a matrix is invertible if and only if when it serves as a coefficient matrix for a system of equations, the system is always consistent independent. It amounts to the second property in Theorem 8.6 where the matrices B are restricted to being $n \times 1$ matrices. We note that, owing to how matrix multiplication is defined, being able to find unique solutions to $AX = B$ for $n \times 1$ matrices B gives you the same statement about solving such equations for $n \times r$ matrices $-$ since we can find a unique solution to them one column at a time.

constants in B. This answers the question as to why we would bother doing row operations on a super-sized augmented matrix to find A^{-1} instead of an ordinary augmented matrix to solve a system; by finding A^{-1} we have done all of the row operations we ever need to do, once and for all, since we can quickly solve *any* equation $AX = B$ using *one* multiplication, $A^{-1}B$.

Example 8.4.1. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$

1. Use row operations to find A^{-1}. Check your answer by finding $A^{-1}A$ and AA^{-1}.

2. Use A^{-1} to solve the following systems of equations

$$\text{(a)} \left\{ \begin{array}{rcl} 3x + y + 2z & = & 26 \\ -y + 5z & = & 39 \\ 2x + y + 4z & = & 117 \end{array} \right. \quad \text{(b)} \left\{ \begin{array}{rcl} 3x + y + 2z & = & 4 \\ -y + 5z & = & 2 \\ 2x + y + 4z & = & 5 \end{array} \right. \quad \text{(c)} \left\{ \begin{array}{rcl} 3x + y + 2z & = & 1 \\ -y + 5z & = & 0 \\ 2x + y + 4z & = & 0 \end{array} \right.$$

Solution.

1. We begin with a super-sized augmented matrix and proceed with Gauss-Jordan elimination.

$$\left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\text{with } \frac{1}{3}R1]{\text{Replace } R1} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow[-2R1 + R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 5 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right] \xrightarrow[\text{with } (-1)R2]{\text{Replace } R2} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right] \xrightarrow[-\frac{1}{3}R2 + R3]{\text{Replace } R3 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & \frac{13}{3} & -\frac{2}{3} & \frac{1}{3} & 1 \end{array} \right] \xrightarrow[\text{with } \frac{3}{13}R3]{\text{Replace } R3} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -5 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right] \xrightarrow[\substack{\text{Replace } R2 \text{ with} \\ 5R3 + R2}]{\substack{\text{Replace } R1 \text{ with} \\ -\frac{2}{3}R3 + R1}} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right]$$

$$\left[\begin{array}{rrr|rrr} 1 & \frac{1}{3} & 0 & \frac{17}{39} & -\frac{2}{39} & -\frac{2}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \xrightarrow[-\frac{1}{3}R2+R1]{\text{Replace } R1 \text{ with}} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ 0 & 1 & 0 & -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$$

We find $A^{-1} = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right]$. To check our answer, we compute

$$A^{-1}A = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right] = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = I_3 \checkmark$$

and

$$AA^{-1} = \left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right] \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = I_3 \checkmark$$

2. Each of the systems in this part has A as its coefficient matrix. The only difference between the systems is the constants which is the matrix B in the associated matrix equation $AX = B$. We solve each of them using the formula $X = A^{-1}B$.

   (a) $X = A^{-1}B = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \left[\begin{array}{r} 26 \\ 39 \\ 117 \end{array}\right] = \left[\begin{array}{r} -39 \\ 91 \\ 26 \end{array}\right]$. Our solution is $(-39, 91, 26)$.

   (b) $X = A^{-1}B = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \left[\begin{array}{r} 4 \\ 2 \\ 5 \end{array}\right] = \left[\begin{array}{r} \frac{5}{13} \\ \frac{19}{13} \\ \frac{9}{13} \end{array}\right]$. We get $\left(\frac{5}{13}, \frac{19}{13}, \frac{9}{13}\right)$.

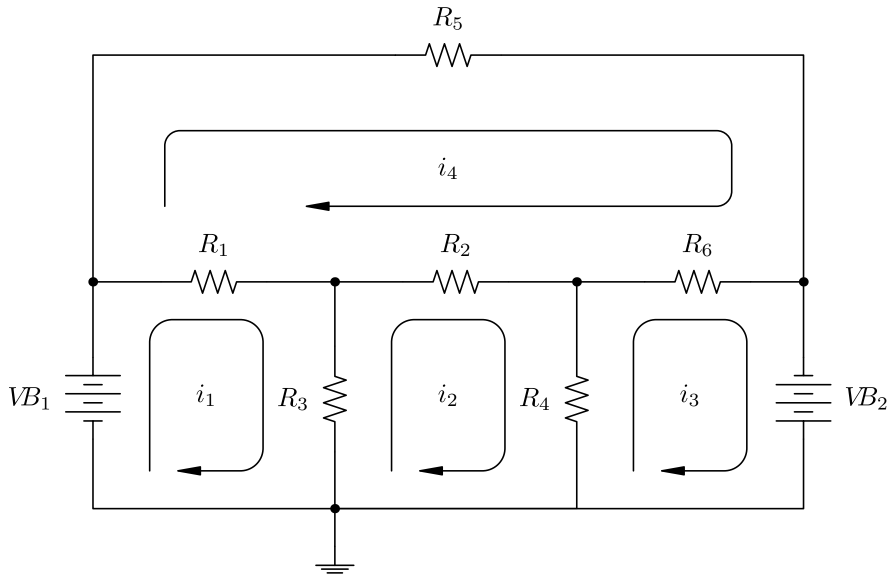
   (c) $X = A^{-1}B = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array}\right] \left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{r} \frac{9}{13} \\ -\frac{10}{13} \\ -\frac{2}{13} \end{array}\right]$. We find $\left(\frac{9}{13}, -\frac{10}{13}, -\frac{2}{13}\right)$.[6]

□

In Example 8.4.1, we see that finding one inverse matrix can enable us to solve an entire family of systems of linear equations. There are many examples of where this comes in handy 'in the wild', and we chose our example for this section from the field of electronics. We also take this opportunity to introduce the student to how we can compute inverse matrices using the calculator.

[6]Note that the solution is the first column of the A^{-1}. The reader is encouraged to meditate on this 'coincidence'.

Example 8.4.2. Consider the circuit diagram below.[7] We have two batteries with source voltages VB_1 and VB_2, measured in volts V, along with six resistors with resistances R_1 through R_6, measured in kiloohms, $k\Omega$. Using Ohm's Law and Kirchhoff's Voltage Law, we can relate the voltage supplied to the circuit by the two batteries to the voltage drops across the six resistors in order to find the four 'mesh' currents: i_1, i_2, i_3 and i_4, measured in milliamps, mA. If we think of electrons flowing through the circuit, we can think of the voltage sources as providing the 'push' which makes the electrons move, the resistors as obstacles for the electrons to overcome, and the mesh current as a net rate of flow of electrons around the indicated loops.



The system of linear equations associated with this circuit is

$$\left\{ \begin{array}{rcl} (R_1 + R_3)\, i_1 - R_3 i_2 - R_1 i_4 & = & VB_1 \\ -R_3 i_1 + (R_2 + R_3 + R_4)\, i_2 - R_4 i_3 - R_2 i_4 & = & 0 \\ -R_4 i_2 + (R_4 + R_6)\, i_3 - R_6 i_4 & = & -VB_2 \\ -R_1 i_1 - R_2 i_2 - R_6 i_3 + (R_1 + R_2 + R_5 + R_6)\, i_4 & = & 0 \end{array} \right.$$

1. Assuming the resistances are all $1k\Omega$, find the mesh currents if the battery voltages are

   (a) $VB_1 = 10V$ and $VB_2 = 5V$

   (b) $VB_1 = 10V$ and $VB_2 = 0V$

   (c) $VB_1 = 0V$ and $VB_2 = 10V$

   (d) $VB_1 = 10V$ and $VB_2 = 10V$

2. Assuming $VB_1 = 10V$ and $VB_2 = 5V$, find the possible combinations of resistances which would yield the mesh currents you found in 1(a).

[7] The authors wish to thank Don Anthan of Lakeland Community College for the design of this example.

Solution.

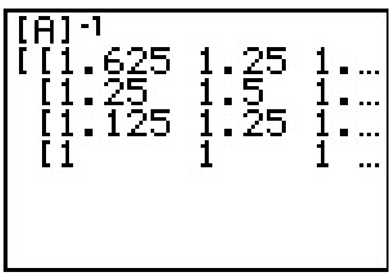
1. Substituting the resistance values into our system of equations, we get

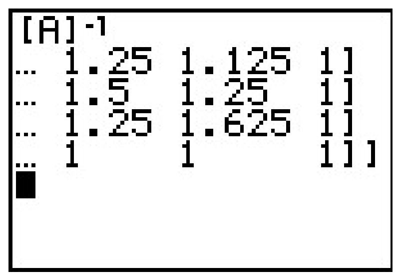
$$\begin{cases} 2i_1 - i_2 - i_4 &= VB_1 \\ -i_1 + 3i_2 - i_3 - i_4 &= 0 \\ -i_2 + 2i_3 - i_4 &= -VB_2 \\ -i_1 - i_2 - i_3 + 4i_4 &= 0 \end{cases}$$

This corresponds to the matrix equation $AX = B$ where

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} \quad B = \begin{bmatrix} VB_1 \\ 0 \\ -VB_2 \\ 0 \end{bmatrix}$$

When we input the matrix A into the calculator, we find



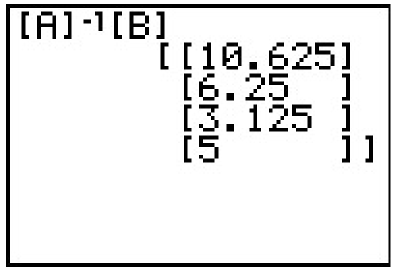


from which we have $A^{-1} = \begin{bmatrix} 1.625 & 1.25 & 1.125 & 1 \\ 1.25 & 1.5 & 1.25 & 1 \\ 1.125 & 1.25 & 1.625 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

To solve the four systems given to us, we find $X = A^{-1}B$ where the value of B is determined by the given values of VB_1 and VB_2

$$1\text{ (a)} \quad B = \begin{bmatrix} 10 \\ 0 \\ -5 \\ 0 \end{bmatrix}, \quad 1\text{ (b)} \quad B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad 1\text{ (c)} \quad B = \begin{bmatrix} 0 \\ 0 \\ -10 \\ 0 \end{bmatrix}, \quad 1\text{ (d)} \quad B = \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

(a) For $VB_1 = 10V$ and $VB_2 = 5V$, the calculator gives $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$, and $i_4 = 5\ mA$. We include a calculator screenshot below for this part (and this part only!) for reference.



(b) By keeping $VB_1 = 10V$ and setting $VB_2 = 0V$, we are removing the effect of the second battery. We get $i_1 = 16.25\ mA$, $i_2 = 12.5\ mA$, $i_3 = 11.25\ mA$, and $i_4 = 10\ mA$.

(c) Part (c) is a symmetric situation to part (b) in so much as we are zeroing out VB_1 and making $VB_2 = 10$. We find $i_1 = -11.25\ mA$, $i_2 = -12.5\ mA$, $i_3 = -16.25\ mA$, and $i_4 = -10\ mA$, where the negatives indicate that the current is flowing in the opposite direction as is indicated on the diagram. The reader is encouraged to study the symmetry here, and if need be, hold up a mirror to the diagram to literally 'see' what is happening.

(d) For $VB_1 = 10V$ and $VB_2 = 10V$, we get $i_1 = 5\ mA$, $i_2 = 0\ mA$, $i_3 = -5\ mA$, and $i_4 = 0\ mA$. The mesh currents i_2 and i_4 being zero is a consequence of both batteries 'pushing' in equal but opposite directions, causing the net flow of electrons in these two regions to cancel out.

2. We now turn the tables and are given $VB_1 = 10V$, $VB_2 = 5V$, $i_1 = 10.625\ mA$, $i_2 = 6.25\ mA$, $i_3 = 3.125\ mA$ and $i_4 = 5\ mA$ and our unknowns are the resistance values. Rewriting our system of equations, we get

$$\left\{ \begin{array}{rcr} 5.625R_1 + 4.375R_3 & = & 10 \\ 1.25R_2 - 4.375R_3 + 3.125R_4 & = & 0 \\ -3.125R_4 - 1.875R_6 & = & -5 \\ -5.625R_1 - 1.25R_2 + 5R_5 + 1.875R_6 & = & 0 \end{array} \right.$$

The coefficient matrix for this system is 4×6 (4 equations with 6 unknowns) and is therefore not invertible. We do know, however, this system is consistent, since setting all the resistance values equal to 1 corresponds to our situation in problem 1a. This means we have an underdetermined consistent system which is necessarily dependent. To solve this system, we encode it into an augmented matrix

$$\left[\begin{array}{rrrrrr|r} 5.25 & 0 & 4.375 & 0 & 0 & 0 & 10 \\ 0 & 1.25 & -4.375 & 3.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.125 & 0 & -1.875 & -5 \\ -5.625 & -1.25 & 0 & 0 & 5 & 1.875 & 0 \end{array} \right]$$

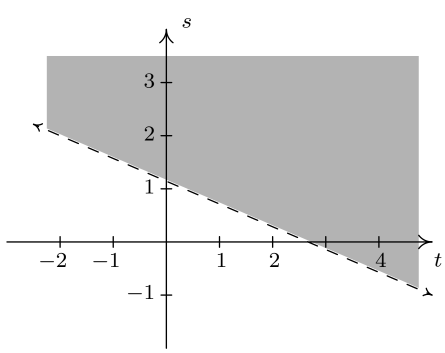
and use the calculator to write in reduced row echelon form

$$\left[\begin{array}{rrrrrr|r} 1 & 0 & 0.\overline{7} & 0 & 0 & 0 & 1.\overline{7} \\ 0 & 1 & -3.5 & 0 & 0 & -1.5 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0.6 & 1.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

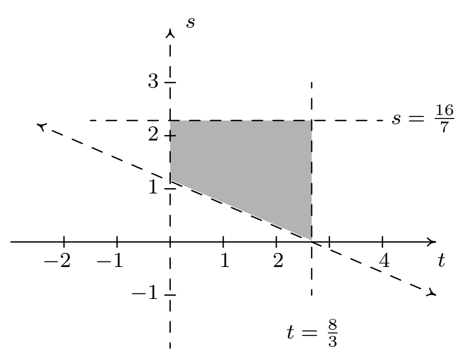
Decoding this system from the matrix, we get

$$\left\{ \begin{array}{rcr} R_1 + 0.\overline{7}R_3 & = & 1.\overline{7} \\ R_2 - 3.5R_3 - 1.5R_6 & = & -4 \\ R_4 + 0.6R_6 & = & 1.6 \\ R_5 & = & 1 \end{array} \right.$$

We have can solve for R_1, R_2, R_4 and R_5 leaving R_3 and R_6 as free variables. Labeling $R_3 = s$ and $R_6 = t$, we have $R_1 = -0.\overline{7}s + 1.\overline{7}$, $R_2 = 3.5s + 1.5t - 4$, $R_4 = -0.6t + 1.6$ and $R_5 = 1$. Since resistance values are always positive, we need to restrict our values of s and t. We know $R_3 = s > 0$ and when we combine that with $R_1 = -0.\overline{7}s + 1.\overline{7} > 0$, we get $0 < s < \frac{16}{7}$. Similarly, $R_6 = t > 0$ and with $R_4 = -0.6t + 1.6 > 0$, we find $0 < t < \frac{8}{3}$. In order visualize the inequality $R_2 = 3.5s + 1.5t - 4 > 0$, we graph the line $3.5s + 1.5t - 4 = 0$ on the st-plane and shade accordingly.[8] Imposing the additional conditions $0 < s < \frac{16}{7}$ and $0 < t < \frac{8}{3}$, we find our values of s and t restricted to the region depicted on the right. Using the roster method, the values of s and t are pulled from the region $\left\{(s,t) : 0 < s < \frac{16}{7},\ 0 < t < \frac{8}{3},\ 3.5s + 1.5t - 4 > 0\right\}$. The reader is encouraged to check that the solution presented in 1(a), namely all resistance values equal to 1, corresponds to a pair (s,t) in the region.



The region where $3.5s + 1.5t - 4 > 0$



The region for our parameters s and t.

□

[8] See Section 2.4 for a review of this procedure.

8.4.1 Exercises

In Exercises 1 - 8, find the inverse of the matrix or state that the matrix is not invertible.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
2. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$
3. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$
4. $D = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$
5. $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$
6. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$
7. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$
8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 11, use one matrix inverse to solve the following systems of linear equations.

9. $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$
10. $\left\{ \begin{array}{rcl} 3x + 7y & = & 0 \\ 5x + 12y & = & -1 \end{array} \right.$
11. $\left\{ \begin{array}{rcl} 3x + 7y & = & -7 \\ 5x + 12y & = & 5 \end{array} \right.$

In Exercises 12 - 14, use the inverse of E from Exercise 5 above to solve the following systems of linear equations.

12. $\left\{ \begin{array}{rcl} 3x + 4z & = & 1 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 0 \end{array} \right.$
13. $\left\{ \begin{array}{rcl} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 1 \\ -3x + 2y - 5z & = & 0 \end{array} \right.$
14. $\left\{ \begin{array}{rcl} 3x + 4z & = & 0 \\ 2x - y + 3z & = & 0 \\ -3x + 2y - 5z & = & 1 \end{array} \right.$

15. This exercise is a continuation of Example 8.3.3 in Section 8.3 and gives another application of matrix inverses. Recall that given the position matrix P for a point in the plane, the matrix RP corresponds to a point rotated 45° counterclockwise from P where

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

   (a) Find R^{-1}.
   (b) If RP rotates a point counterclockwise 45°, what should $R^{-1}P$ do? Check your answer by finding $R^{-1}P$ for various points on the coordinate axes and the lines $y = \pm x$.
   (c) Find $R^{-1}P$ where P corresponds to a generic point $P(x, y)$. Verify that this takes points on the curve $y = \frac{2}{x}$ to points on the curve $x^2 - y^2 = 4$.

16. A Sasquatch's diet consists of three primary foods: Ippizuti Fish, Misty Mushrooms, and Sun Berries. Each serving of Ippizuti Fish is 500 calories, contains 40 grams of protein, and has no Vitamin X. Each serving of Misty Mushrooms is 50 calories, contains 1 gram of protein, and 5 milligrams of Vitamin X. Finally, each serving of Sun Berries is 80 calories, contains no protein, but has 15 milligrams of Vitamin X.[9]

    (a) If an adult male Sasquatch requires 3200 calories, 130 grams of protein, and 275 milligrams of Vitamin X daily, use a matrix inverse to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries he needs to eat each day.

    (b) An adult female Sasquatch requires 3100 calories, 120 grams of protein, and 300 milligrams of Vitamin X daily. Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

    (c) An adolescent Sasquatch requires 5000 calories, 400 grams of protein daily, but no Vitamin X daily.[10] Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

17. Matrices can be used in cryptography. Suppose we wish to encode the message 'BIGFOOT LIVES'. We start by assigning a number to each letter of the alphabet, say $A = 1$, $B = 2$ and so on. We reserve 0 to act as a space. Hence, our message 'BIGFOOT LIVES' corresponds to the string of numbers '2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19.' To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

Since A is 3×3 matrix, we encode our message string into a matrix M with 3 rows. To do this, we take the first three numbers, 2 9 7, and make them our first column, the next three numbers, 6 15 15, and make them our second column, and so on. We put 0's to round out the matrix.

$$M = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

To encode the message, we find the product AM

$$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

[9] Misty Mushrooms and Sun Berries are the only known fictional sources of Vitamin X.

[10] Vitamin X is needed to sustain Sasquatch longevity only.

So our coded message is '12, 1, −12, 42, 3, −42, 100, 36, −152, −23, 39, −46, 38, 57, −133.' To decode this message, we start with this string of numbers, construct a message matrix as we did earlier (we should get the matrix AM again) and then multiply by A^{-1}.

(a) Find A^{-1}.

(b) Use A^{-1} to decode the message and check this method actually works.

(c) Decode the message '14, 37, −76, 128, 21, −151, 31, 65, −140'

(d) Choose another invertible matrix and encode and decode your own messages.

18. Using the matrices A from Exercise 1, B from Exercise 2 and D from Exercise 4, show $AB = D$ and $D^{-1} = B^{-1}A^{-1}$. That is, show that $(AB)^{-1} = B^{-1}A^{-1}$.

19. Let M and N be invertible $n \times n$ matrices. Show that $(MN)^{-1} = N^{-1}M^{-1}$ and compare your work to Exercise 31 in Section 5.2.

8.4.2 Answers

1. $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

2. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

3. C is not invertible

4. $D^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{1}{2} \\ 8 & -1 \end{bmatrix}$

5. $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

6. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

7. G is not invertible

8. $H^{-1} = \begin{bmatrix} 16 & 0 & 3 & 0 \\ -90 & -\frac{1}{2} & -\frac{35}{2} & \frac{7}{2} \\ 5 & 0 & 1 & 0 \\ -36 & 0 & -7 & 1 \end{bmatrix}$

The coefficient matrix is B^{-1} from Exercise 2 above so the inverse we need is $(B^{-1})^{-1} = B$.

9. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 26 \\ 39 \end{bmatrix} = \begin{bmatrix} 39 \\ -13 \end{bmatrix}$ So $x = 39$ and $y = -13$.

10. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ So $x = 7$ and $y = -3$.

11. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -119 \\ 50 \end{bmatrix}$ So $x = -119$ and $y = 50$.

The coefficient matrix is $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$ from Exercise 5, so $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ So $x = -1$, $y = 1$ and $z = 1$.

13. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -6 \end{bmatrix}$ So $x = 8$, $y = -3$ and $z = -6$.

14. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$ So $x = 4$, $y = -1$ and $z = -3$.

16. (a) The adult male Sasquatch needs: 3 servings of Ippizuti Fish, 10 servings of Misty Mushrooms, and 15 servings of Sun Berries daily.

    (b) The adult female Sasquatch needs: 3 servings of Ippizuti Fish and 20 servings of Sun Berries daily. (No Misty Mushrooms are needed!)

    (c) The adolescent Sasquatch requires 10 servings of Ippizuti Fish daily. (No Misty Mushrooms or Sun Berries are needed!)

17. (a) $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix} \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$ ✓

    (c) 'LOGS RULE'

8.5 Determinants and Cramer's Rule

8.5.1 Definition and Properties of the Determinant

In this section we assign to each square matrix A a real number, called the **determinant** of A, which will eventually lead us to yet another technique for solving consistent independent systems of linear equations. The determinant is defined recursively, that is, we define it for 1×1 matrices and give a rule by which we can reduce determinants of $n \times n$ matrices to a sum of determinants of $(n-1) \times (n-1)$ matrices.[1] This means we will be able to evaluate the determinant of a 2×2 matrix as a sum of the determinants of 1×1 matrices; the determinant of a 3×3 matrix as a sum of the determinants of 2×2 matrices, and so forth. To explain how we will take an $n \times n$ matrix and distill from it an $(n-1) \times (n-1)$, we use the following notation.

Definition 8.12. Given an $n \times n$ matrix A where $n > 1$, the matrix A_{ij} is the $(n-1) \times (n-1)$ matrix formed by deleting the ith row of A and the jth column of A.

For example, using the matrix A below, we find the matrix A_{23} by deleting the second row and third column of A.

$$A = \left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array} \right] \xrightarrow{\text{Delete } R2 \text{ and } C3} A_{23} = \left[\begin{array}{rr} 3 & 1 \\ 2 & 1 \end{array} \right]$$

We are now in the position to define the determinant of a matrix.

Definition 8.13. Given an $n \times n$ matrix A the **determinant of A**, denoted $\det(A)$, is defined as follows

- If $n = 1$, then $A = [a_{11}]$ and $\det(A) = \det\left([a_{11}]\right) = a_{11}$.
- If $n > 1$, then $A = [a_{ij}]_{n \times n}$ and

$$\det(A) = \det\left([a_{ij}]_{n \times n}\right) = a_{11}\det\left(A_{11}\right) - a_{12}\det\left(A_{12}\right) + - \ldots + (-1)^{1+n} a_{1n} \det\left(A_{1n}\right)$$

There are two commonly used notations for the determinant of a matrix A: '$\det(A)$' and '$|A|$' We have chosen to use the notation $\det(A)$ as opposed to $|A|$ because we find that the latter is often confused with absolute value, especially in the context of a 1×1 matrix. In the expansion $a_{11}\det\left(A_{11}\right) - a_{12}\det\left(A_{12}\right) + - \ldots + (-1)^{1+n} a_{1n} \det\left(A_{1n}\right)$, the notation '$+ - \ldots + (-1)^{1+n} a_{1n}$' means that the signs alternate and the final sign is dictated by the sign of the quantity $(-1)^{1+n}$. Since the entries a_{11}, a_{12} and so forth up through a_{1n} comprise the first row of A, we say we are finding the determinant of A by 'expanding along the first row'. Later in the section, we will develop a formula for $\det(A)$ which allows us to find it by expanding along any row.

Applying Definition 8.13 to the matrix $A = \left[\begin{array}{rr} 4 & -3 \\ 2 & 1 \end{array} \right]$ we get

[1]We will talk more about the term 'recursively' in Section 9.1.

$$\begin{aligned}\det(A) &= \det\left(\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\right) \\ &= 4\det\left(A_{11}\right) - (-3)\det\left(A_{12}\right) \\ &= 4\det([1]) + 3\det([2]) \\ &= 4(1) + 3(2) \\ &= 10\end{aligned}$$

For a generic 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

$$\begin{aligned}\det(A) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= a\det\left(A_{11}\right) - b\det\left(A_{12}\right) \\ &= a\det\left([d]\right) - b\det\left([c]\right) \\ &= ad - bc\end{aligned}$$

This formula is worth remembering

Equation 8.1. For a 2×2 matrix,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Applying Definition 8.13 to the 3×3 matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ we obtain

$$\begin{aligned}\det(A) &= \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) \\ &= 3\det\left(A_{11}\right) - 1\det\left(A_{12}\right) + 2\det\left(A_{13}\right) \\ &= 3\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\right) \\ &= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) \\ &= 3(-9) - (-10) + 2(2) \\ &= -13\end{aligned}$$

To evaluate the determinant of a 4×4 matrix, we would have to evaluate the determinants of *four* 3×3 matrices, each of which involves the finding the determinants of *three* 2×2 matrices. As you can see, our method of evaluating determinants quickly gets out of hand and many of you may be reaching for the calculator. There is some mathematical machinery which can assist us in calculating determinants and we present that here. Before we state the theorem, we need some more terminology.

Definition 8.14. Let A be an $n \times n$ matrix and A_{ij} be defined as in Definition 8.12. The ***ij* minor** of A, denoted M_{ij} is defined by $M_{ij} = \det(A_{ij})$. The ***ij* cofactor** of A, denoted C_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij})$.

We note that in Definition 8.13, the sum

$$a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

can be rewritten as

$$a_{11}(-1)^{1+1} \det(A_{11}) + a_{12}(-1)^{1+2} \det(A_{12}) + \ldots + a_{1n}(-1)^{1+n} \det(A_{1n})$$

which, in the language of cofactors is

$$a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

We are now ready to state our main theorem concerning determinants.

Theorem 8.7. Properties of the Determinant: Let $A = [a_{ij}]_{n \times n}$.

- We may find the determinant by expanding along any row. That is, for any $1 \leq k \leq n$,

$$\det(A) = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$

- If A' is the matrix obtained from A by:
  - interchanging any two rows, then $\det(A') = -\det(A)$.
  - replacing a row with a nonzero multiple (say c) of itself, then $\det(A') = c\det(A)$
  - replacing a row with itself plus a multiple of another row, then $\det(A') = \det(A)$
- If A has two identical rows, or a row consisting of all 0's, then $\det(A) = 0$.
- If A is upper or lower triangular,[a] then $\det(A)$ is the product of the entries on the main diagonal.[b]
- If B is an $n \times n$ matrix, then $\det(AB) = \det(A)\det(B)$.
- $\det(A^n) = \det(A)^n$ for all natural numbers n.
- A is invertible if and only if $\det(A) \neq 0$. In this case, $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$.

[a]See Exercise 8.3.1 in 8.3.
[b]See page 585 in Section 8.3.

Unfortunately, while we can easily *demonstrate* the results in Theorem 8.7, the proofs of most of these properties are beyond the scope of this text. We could prove these properties for generic 2×2

or even 3×3 matrices by brute force computation, but this manner of proof belies the elegance and symmetry of the determinant. We will prove what few properties we can after we have developed some more tools such as the Principle of Mathematical Induction in Section 9.3.[2] For the moment, let us demonstrate some of the properties listed in Theorem 8.7 on the matrix A below. (Others will be discussed in the Exercises.)

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found $\det(A) = -13$ by expanding along the first row. To take advantage of the 0 in the second row, we use Theorem 8.7to find $\det(A) = -13$ by expanding along that row.

$$\begin{aligned}
\det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) &= 0C_{21} + (-1)C_{22} + 5C_{23} \\
&= (-1)(-1)^{2+2}\det\left(A_{22}\right) + 5(-1)^{2+3}\det\left(A_{23}\right) \\
&= -\det\left(\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\right) - 5\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right) \\
&= -((3)(4) - (2)(2)) - 5((3)(1) - (2)(1)) \\
&= -8 - 5 \\
&= -13 \checkmark
\end{aligned}$$

In general, the sign of $(-1)^{i+j}$ in front of the minor in the expansion of the determinant follows an alternating pattern. Below is the pattern for 2×2, 3×3 and 4×4 matrices, and it extends naturally to higher dimensions.

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

The reader is cautioned, however, against reading too much into these sign patterns. In the example above, we expanded the 3×3 matrix A by its second row and the term which corresponds to the second entry ended up being negative even though the sign attached to the minor is $(+)$. These signs represent only the signs of the $(-1)^{i+j}$ in the formula; the sign of the corresponding entry as well as the minor itself determine the ultimate sign of the term in the expansion of the determinant.

To illustrate some of the other properties in Theorem 8.7, we use row operations to transform our 3×3 matrix A into an upper triangular matrix, keeping track of the row operations, and labeling

[2] For a very elegant treatment, take a course in Linear Algebra. There, you will most likely see the treatment of determinants logically reversed than what is presented here. Specifically, the determinant is defined as a function which takes a square matrix to a real number and satisfies some of the properties in Theorem 8.7. From that function, a formula for the determinant is developed.

each successive matrix.[3]

$$\underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right]}_{A} \xrightarrow[\text{with } -\frac{2}{3}R1+R3]{\text{Replace } R3} \underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & \frac{1}{3} & \frac{8}{3} \end{array}\right]}_{B} \xrightarrow[\frac{1}{3}R2+R3]{\text{Replace } R3 \text{ with}} \underbrace{\left[\begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & \frac{13}{3} \end{array}\right]}_{C}$$

Theorem 8.7 guarantees us that $\det(A) = \det(B) = \det(C)$ since we are replacing a row with itself plus a multiple of another row moving from one matrix to the next. Furthermore, since C is upper triangular, $\det(C)$ is the product of the entries on the main diagonal, in this case $\det(C) = (3)(-1)\left(\frac{13}{3}\right) = -13$. This demonstrates the utility of using row operations to assist in calculating determinants. This also sheds some light on the connection between a determinant and invertibility. Recall from Section 8.4 that in order to find A^{-1}, we attempt to transform A to I_n using row operations

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

As we apply our allowable row operations on A to put it into reduced row echelon form, the determinant of the intermediate matrices can vary from the determinant of A by at most a *nonzero* multiple. This means that if $\det(A) \neq 0$, then the determinant of A's reduced row echelon form must also be nonzero, which, according to Definition 8.4 means that all the main diagonal entries on A's reduced row echelon form must be 1. That is, A's reduced row echelon form is I_n, and A is invertible. Conversely, if A is invertible, then A can be transformed into I_n using row operations. Since $\det(I_n) = 1 \neq 0$, our same logic implies $\det(A) \neq 0$. Basically, we have established that the determinant *determines* whether or not the matrix A is invertible.[4]

It is worth noting that when we first introduced the notion of a matrix inverse, it was in the context of solving a linear matrix equation. In effect, we were trying to 'divide' both sides of the matrix equation $AX = B$ by the matrix A. Just like we cannot divide a real number by 0, Theorem 8.7 tells us we cannot 'divide' by a matrix whose *determinant* is 0. We also know that if the coefficient matrix of a system of linear equations is invertible, then system is consistent and independent. It follows, then, that if the determinant of said coefficient is not zero, the system is consistent and independent.

8.5.2 Cramer's Rule and Matrix Adjoints

In this section, we introduce a theorem which enables us to solve a system of linear equations by means of determinants only. As usual, the theorem is stated in full generality, using numbered unknowns x_1, x_2, etc., instead of the more familiar letters x, y, z, etc. The proof of the general case is best left to a course in Linear Algebra.

[3]Essentially, we follow the Gauss Jordan algorithm but we don't care about getting leading 1's.

[4]In Section 8.5.2, we learn determinants (specifically cofactors) are deeply connected with the inverse of a matrix.

Theorem 8.8. Cramer's Rule: Suppose $AX = B$ is the matrix form of a system of n linear equations in n unknowns where A is the coefficient matrix, X is the unknowns matrix, and B is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1, x_2, \ldots x_n$ is given by:

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where A_j is the matrix A whose jth column has been replaced by the constants in B.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of A_j to that of the determinant of the coefficient matrix. The matrix A_j is found by replacing the column in the coefficient matrix which holds the coefficients of x_j with the constants of the system. The following example fleshes out this method.

Example 8.5.1. Use Cramer's Rule to solve for the indicated unknowns.

1. Solve $\left\{ \begin{array}{rcr} 2x_1 - 3x_2 & = & 4 \\ 5x_1 + x_2 & = & -2 \end{array} \right.$ for x_1 and x_2

2. Solve $\left\{ \begin{array}{rcr} 2x - 3y + z & = & -1 \\ x - y + z & = & 1 \\ 3x - 4z & = & 0 \end{array} \right.$ for z.

Solution.

1. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

To find the matrix A_1, we remove the column of the coefficient matrix A which holds the coefficients of x_1 and replace it with the corresponding entries in B. Likewise, we replace the column of A which corresponds to the coefficients of x_2 with the constants to form the matrix A_2. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$ and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

2. To use Cramer's Rule to find z, we identify x_3 as z. We have

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A_3 = A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Expanding both $\det(A)$ and $\det(A_z)$ along the third rows (to take advantage of the 0's) gives

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$$

The reader is encouraged to solve this system for x and y similarly and check the answer. □

Our last application of determinants is to develop an alternative method for finding the inverse of a matrix.[5] Let us consider the 3×3 matrix A which we so extensively studied in Section 8.5.1

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found through a variety of methods that $\det(A) = -13$. To our surprise and delight, its inverse below has a remarkable number of 13's in the denominators of its entries. This is no coincidence.

$$A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$$

Recall that to find A^{-1}, we are essentially solving the matrix equation $AX = I_3$, where $X = [x_{ij}]_{3\times 3}$ is a 3×3 matrix. Because of how matrix multiplication is defined, the first column of I_3 is the product of A with the first column of X, the second column of I_3 is the product of A with the second column of X and the third column of I_3 is the product of A with the third column of X. In other words, we are solving three equations[6]

$$A \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can solve each of these systems using Cramer's Rule. Focusing on the first system, we have

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 5 \\ 2 & 0 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

[5] We are developing a *method* in the forthcoming discussion. As with the discussion in Section 8.4 when we developed the first algorithm to find matrix inverses, we ask that you indulge us.

[6] The reader is encouraged to stop and think this through.

If we expand $\det(A_1)$ along the first row, we get

$$\begin{aligned}\det(A_1) &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right)\end{aligned}$$

Amazingly, this is none other than the C_{11} cofactor of A. The reader is invited to check this, as well as the claims that $\det(A_2) = C_{12}$ and $\det(A_3) = C_{13}$.[7] (To see this, though it seems unnatural to do so, expand along the first row.) Cramer's Rule tells us

$$x_{11} = \frac{\det(A_1)}{\det(A)} = \frac{C_{11}}{\det(A)}, \quad x_{21} = \frac{\det(A_2)}{\det(A)} = \frac{C_{12}}{\det(A)}, \quad x_{31} = \frac{\det(A_3)}{\det(A)} = \frac{C_{13}}{\det(A)}$$

So the first column of the inverse matrix X is:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{\det(A)} \\ \dfrac{C_{12}}{\det(A)} \\ \dfrac{C_{13}}{\det(A)} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

Notice the reversal of the subscripts going from the unknown to the corresponding cofactor of A. This trend continues and we get

$$\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} \qquad \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix}$$

Putting all of these together, we have obtained a new and surprising formula for A^{-1}, namely

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

To see that this does indeed yield A^{-1}, we find all of the cofactors of A

$$\begin{array}{rclrclrcl} C_{11} & = & -9, & C_{21} & = & -2, & C_{31} & = & 7 \\ C_{12} & = & 10, & C_{22} & = & 8, & C_{32} & = & -15 \\ C_{13} & = & 2, & C_{23} & = & -1, & C_{33} & = & -3 \end{array}$$

And, as promised,

[7] In a solid Linear Algebra course you will learn that the properties in Theorem 8.7 hold equally well if the word 'row' is replaced by the word 'column'. We're not going to get into column operations in this text, but they do make some of what we're trying to say easier to follow.

$$A^{-1} = \frac{1}{\det(A)} \left[\begin{array}{ccc} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{array} \right] = -\frac{1}{13} \left[\begin{array}{rrr} -9 & -2 & 7 \\ 10 & 8 & -15 \\ 2 & -1 & -3 \end{array} \right] = \left[\begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right]$$

To generalize this to invertible $n \times n$ matrices, we need another definition and a theorem. Our definition gives a special name to the cofactor matrix, and the theorem tells us how to use it along with $\det(A)$ to find the inverse of a matrix.

Definition 8.15. Let A be an $n \times n$ matrix, and C_{ij} denote the ij cofactor of A. The **adjoint** of A, denoted $\text{adj}(A)$ is the matrix whose ij-entry is the ji cofactor of A, C_{ji}. That is

$$\text{adj}(A) = \left[\begin{array}{cccc} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{array} \right]$$

This new notation greatly shortens the statement of the formula for the inverse of a matrix.

Theorem 8.9. Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

For 2×2 matrices, Theorem 8.9 reduces to a fairly simple formula.

Equation 8.2. For an invertible 2×2 matrix,

$$\left[\begin{array}{cc} a & b \\ c & d \end{array} \right]^{-1} = \frac{1}{ad - bc} \left[\begin{array}{rr} d & -b \\ -c & a \end{array} \right]$$

The proof of Theorem 8.9 is, like so many of the results in this section, best left to a course in Linear Algebra. In such a course, not only do you gain some more sophisticated proof techniques, you also gain a larger perspective. The authors assure you that persistence pays off. If you stick around a few semesters and take a course in Linear Algebra, you'll see just how pretty all things matrix really are - in spite of the tedious notation and sea of subscripts. Within the scope of this text, we will prove a few results involving determinants in Section 9.3 once we have the Principle of Mathematical Induction well in hand. Until then, make sure you have a handle on the *mechanics* of matrices and the theory will come eventually.

8.5.3 Exercises

In Exercises 1 - 8, compute the determinant of the given matrix. (Some of these matrices appeared in Exercises 1 - 8 in Section 8.4.)

1. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

2. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

3. $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

4. $L = \begin{bmatrix} \dfrac{1}{x^3} & \dfrac{\ln(x)}{x^3} \\ -\dfrac{3}{x^4} & \dfrac{1-3\ln(x)}{x^4} \end{bmatrix}$

5. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

6. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

7. $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 14, use Cramer's Rule to solve the system of linear equations.

9. $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$

10. $\left\{ \begin{array}{rcr} 2x - 4y & = & 5 \\ 10x + 13y & = & -6 \end{array} \right.$

11. $\left\{ \begin{array}{rcr} x + y & = & 8000 \\ 0.03x + 0.05y & = & 250 \end{array} \right.$

12. $\left\{ \begin{array}{rcl} \frac{1}{2}x - \frac{1}{5}y & = & 1 \\ 6x + 7y & = & 3 \end{array} \right.$

13. $\left\{ \begin{array}{rcl} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} 3x + y - 2z & = & 10 \\ 4x - y + z & = & 5 \\ x - 3y - 4z & = & -1 \end{array} \right.$

In Exercises 15 - 16, use Cramer's Rule to solve for x_4.

15. $\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} 4x_1 + x_2 & = & 4 \\ x_2 - 3x_3 & = & 1 \\ 10x_1 + x_3 + x_4 & = & 0 \\ -x_2 + x_3 & = & -3 \end{array} \right.$

In Exercises 17 - 18, find the inverse of the given matrix using their determinants and adjoints.

17. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

18. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

19. Carl's Sasquatch Attack! Game Card Collection is a mixture of common and rare cards. Each common card is worth \$0.25 while each rare card is worth \$0.75. If his entire 117 card collection is worth \$48.75, how many of each kind of card does he own?

20. How much of a 5 gallon 40% salt solution should be replaced with pure water to obtain 5 gallons of a 15% solution?

21. How much of a 10 liter 30% acid solution must be replaced with pure acid to obtain 10 liters of a 50% solution?

22. Daniel's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind he boards, Daniel answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: tarantulas have 8 legs and no tails, scorpions have 8 legs and one tail, and snakes have no legs and one tail.)

23. This exercise is a continuation of Exercise 16 in Section 8.4. Just because a system is consistent independent doesn't mean it will admit a solution that makes sense in an applied setting. Using the nutrient values given for Ippizuti Fish, Misty Mushrooms, and Sun Berries, use Cramer's Rule to determine the number of servings of Ippizuti Fish needed to meet the needs of a daily diet which requires 2500 calories, 1000 grams of protein, and 400 milligrams of Vitamin X. Now use Cramer's Rule to find the number of servings of Misty Mushrooms required. Does a solution to this diet problem exist?

24. Let $R = \begin{bmatrix} -7 & 3 \\ 11 & 2 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -5 \\ 6 & 9 \end{bmatrix}$ $T = \begin{bmatrix} 11 & 2 \\ -7 & 3 \end{bmatrix}$, and $U = \begin{bmatrix} -3 & 15 \\ 6 & 9 \end{bmatrix}$

    (a) Show that $\det(RS) = \det(R)\det(S)$

    (b) Show that $\det(T) = -\det(R)$

    (c) Show that $\det(U) = -3\det(S)$

25. For M, N, and P below, show that $\det(M) = 0$, $\det(N) = 0$ and $\det(P) = 0$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

26. Let A be an arbitrary invertible 3×3 matrix.

 (a) Show that $\det(I_3) = 1$.[8]

 (b) Using the facts that $AA^{-1} = I_3$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

The purpose of Exercises 27 - 30 is to introduce you to the eigenvalues and eigenvectors of a matrix.[9] We begin with an example using a 2×2 matrix and then guide you through some exercises using a 3×3 matrix. Consider the matrix

$$C = \left[\begin{array}{rr} 6 & 15 \\ 14 & 35 \end{array} \right]$$

from Exercise 2. We know that $\det(C) = 0$ which means that $CX = 0_{2\times 2}$ does not have a unique solution. So there is a nonzero matrix Y with $CY = 0_{2\times 2}$. In fact, every matrix of the form

$$Y = \left[\begin{array}{r} -\frac{5}{2}t \\ t \end{array} \right]$$

is a solution to $CX = 0_{2\times 2}$, so there are infinitely many matrices such that $CX = 0_{2\times 2}$. But consider the matrix

$$X_{41} = \left[\begin{array}{r} 3 \\ 7 \end{array} \right]$$

It is NOT a solution to $CX = 0_{2\times 2}$, but rather,

$$CX_{41} = \left[\begin{array}{rr} 6 & 15 \\ 14 & 35 \end{array} \right] \left[\begin{array}{r} 3 \\ 7 \end{array} \right] = \left[\begin{array}{r} 123 \\ 287 \end{array} \right] = 41 \left[\begin{array}{r} 3 \\ 7 \end{array} \right]$$

In fact, if Z is of the form

$$Z = \left[\begin{array}{r} \frac{3}{7}t \\ t \end{array} \right]$$

then

$$CZ = \left[\begin{array}{rr} 6 & 15 \\ 14 & 35 \end{array} \right] \left[\begin{array}{r} \frac{3}{7}t \\ t \end{array} \right] = \left[\begin{array}{r} \frac{123}{7}t \\ 41t \end{array} \right] = 41 \left[\begin{array}{r} \frac{3}{7}t \\ t \end{array} \right] = 41Z$$

for all t. The big question is "How did we know to use 41?"

We need a number λ such that $CX = \lambda X$ has nonzero solutions. We have demonstrated that $\lambda = 0$ and $\lambda = 41$ both worked. Are there others? If we look at the matrix equation more closely, what

[8] If you think about it for just a moment, you'll see that $\det(I_n) = 1$ for any natural number n. The formal proof of this fact requires the Principle of Mathematical Induction (Section 9.3) so we'll stick with $n = 3$ for the time being.

[9] This material is usually given its own chapter in a Linear Algebra book so clearly we're not able to tell you everything you need to know about eigenvalues and eigenvectors. They are a nice application of determinants, though, so we're going to give you enough background so that you can start playing around with them.

we *really* wanted was a nonzero solution to $(C - \lambda I_2)X = 0_{2\times 2}$ which we know exists if and only if the determinant of $C - \lambda I_2$ is zero.[10] So we computed

$$\det(C - \lambda I_2) = \det\left(\begin{bmatrix} 6-\lambda & 15 \\ 14 & 35-\lambda \end{bmatrix}\right) = (6-\lambda)(35-\lambda) - 14\cdot 15 = \lambda^2 - 41\lambda$$

This is called the **characteristic polynomial** of the matrix C and it has two zeros: $\lambda = 0$ and $\lambda = 41$. That's how we knew to use 41 in our work above. The fact that $\lambda = 0$ showed up as one of the zeros of the characteristic polynomial just means that C itself had determinant zero which we already knew. Those two numbers are called the **eigenvalues** of C. The corresponding matrix solutions to $CX = \lambda X$ are called the **eigenvectors** of C and the 'vector' portion of the name will make more sense after you've studied vectors.

Now it's your turn. In the following exercises, you'll be using the matrix G from Exercise 6.

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$$

27. Show that the characteristic polynomial of G is $p(\lambda) = -\lambda(\lambda - 1)(\lambda - 22)$. That is, compute $\det(G - \lambda I_3)$.

28. Let $G_0 = G$. Find the parametric description of the solution to the system of linear equations given by $GX = 0_{3\times 3}$.

29. Let $G_1 = G - I_3$. Find the parametric description of the solution to the system of linear equations given by $G_1 X = 0_{3\times 3}$. Show that any solution to $G_1 X = 0_{3\times 3}$ also has the property that $GX = 1X$.

30. Let $G_{22} = G - 22I_3$. Find the parametric description of the solution to the system of linear equations given by $G_{22}X = 0_{3\times 3}$. Show that any solution to $G_{22}X = 0_{3\times 3}$ also has the property that $GX = 22X$.

[10] Think about this.

8.5.4 Answers

1. $\det(B) = 1$
2. $\det(C) = 0$
3. $\det(Q) = x^2$
4. $\det(L) = \dfrac{1}{x^7}$
5. $\det(F) = -12$
6. $\det(G) = 0$
7. $\det(V) = 20i + 43j + 4k$
8. $\det(H) = -2$
9. $x = 39,\ y = -13$
10. $x = \frac{41}{66},\ y = -\frac{31}{33}$
11. $x = 7500,\ y = 500$
12. $x = \frac{76}{47},\ y = -\frac{45}{47}$
13. $x = 1,\ y = 2,\ z = 0$
14. $x = \frac{121}{60},\ y = \frac{131}{60},\ z = -\frac{53}{60}$
15. $x_4 = 4$
16. $x_4 = -1$
17. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$
18. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$
19. Carl owns 78 common cards and 39 rare cards.
20. 3.125 gallons.
21. $\frac{20}{7} \approx 2.85$ liters.
22. The rescue houses 15 snakes, 21 tarantulas and 13 scorpions.
23. Using Cramer's Rule, we find we need 53 servings of Ippizuti Fish to satisfy the dietary requirements. The number of servings of Misty Mushrooms required, however, is -1120. Since it's impossible to have a negative number of servings, there is no solution to the applied problem, despite there being a solution to the mathematical problem. A cautionary tale about using Cramer's Rule: just because you are guaranteed a mathematical answer for each variable doesn't mean the solution will make sense in the 'real' world.

8.6 Partial Fraction Decomposition

This section uses systems of linear equations to rewrite rational functions in a form more palatable to Calculus students. In College Algebra, the function

$$f(x) = \frac{x^2 - x - 6}{x^4 + x^2} \tag{1}$$

is written in the best form possible to construct a sign diagram and to find zeros and asymptotes, but certain applications in Calculus require us to rewrite $f(x)$ as

$$f(x) = \frac{x+7}{x^2+1} - \frac{1}{x} - \frac{6}{x^2} \tag{2}$$

If we are given the form of $f(x)$ in (2), it is a matter of Intermediate Algebra to determine a common denominator to obtain the form of $f(x)$ given in (1). The focus of this section is to develop a method by which we start with $f(x)$ in the form of (1) and 'resolve it into **partial fractions**' to obtain the form in (2). Essentially, we need to reverse the least common denominator process. Starting with the form of $f(x)$ in (1), we begin by factoring the denominator

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2+1\right)}$$

We now think about which individual denominators could contribute to obtain $x^2\left(x^2+1\right)$ as the least common denominator. Certainly x^2 and x^2+1, but are there any other factors? Since x^2+1 is an irreducible quadratic[1] there are no factors of it that have real coefficients which can contribute to the denominator. The factor x^2, however, is not irreducible, since we can think of it as $x^2 = xx = (x-0)(x-0)$, a so-called 'repeated' linear factor.[2] This means it's possible that a term with a denominator of just x contributed to the expression as well. What about something like $x\left(x^2+1\right)$? This, too, could contribute, but we would then wish to break down that denominator into x and $\left(x^2+1\right)$, so we leave out a term of that form. At this stage, we have guessed

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2+1\right)} = \frac{?}{x} + \frac{?}{x^2} + \frac{?}{x^2+1}$$

Our next task is to determine what form the unknown numerators take. It stands to reason that since the expression $\frac{x^2-x-6}{x^4+x^2}$ is 'proper' in the sense that the degree of the numerator is less than the degree of the denominator, we are safe to make the ansatz that all of the partial fraction resolvents are also. This means that the numerator of the fraction with x as its denominator is just a constant and the numerators on the terms involving the denominators x^2 and x^2+1 are at most linear polynomials. That is, we guess that there are real numbers A, B, C, D and E so that

$$\frac{x^2 - x - 6}{x^4 + x^2} = \frac{x^2 - x - 6}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2} + \frac{Dx+E}{x^2+1}$$

[1] Recall this means it has no real zeros; see Section 3.4.

[2] Recall this means $x = 0$ is a zero of multiplicity 2.

However, if we look more closely at the term $\frac{Bx+C}{x^2}$, we see that $\frac{Bx+C}{x^2} = \frac{Bx}{x^2} + \frac{C}{x^2} = \frac{B}{x} + \frac{C}{x^2}$. The term $\frac{B}{x}$ has the same form as the term $\frac{A}{x}$ which means it contributes nothing new to our expansion. Hence, we drop it and, after re-labeling, we find ourselves with our new guess:

$$\frac{x^2-x-6}{x^4+x^2} = \frac{x^2-x-6}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Our next task is to determine the values of our unknowns. Clearing denominators gives

$$x^2-x-6 = Ax\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)x^2$$

Gathering the like powers of x we have

$$x^2-x-6 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

In order for this to hold for all values of x in the domain of f, we equate the coefficients of corresponding powers of x on each side of the equation[3] and obtain the system of linear equations

$$\left\{ \begin{array}{llrl} (E1) & A+C & = & 0 \quad \text{From equating coefficients of } x^3 \\ (E2) & B+D & = & 1 \quad \text{From equating coefficients of } x^2 \\ (E3) & A & = & -1 \quad \text{From equating coefficients of } x \\ (E4) & B & = & -6 \quad \text{From equating the constant terms} \end{array} \right.$$

To solve this system of equations, we could use any of the methods presented in Sections 8.1 through 8.5, but none of these methods are as efficient as the good old-fashioned substitution you learned in Intermediate Algebra. From $E3$, we have $A=-1$ and we substitute this into $E1$ to get $C=1$. Similarly, since $E4$ gives us $B=-6$, we have from $E2$ that $D=7$. We get

$$\frac{x^2-x-6}{x^4+x^2} = \frac{x^2-x-6}{x^2\left(x^2+1\right)} = -\frac{1}{x} - \frac{6}{x^2} + \frac{x+7}{x^2+1}$$

which matches the formula given in (2). As we have seen in this opening example, resolving a rational function into partial fractions takes two steps: first, we need to determine the *form* of the decomposition, and then we need to determine the unknown coefficients which appear in said form. Theorem 3.16 guarantees that any polynomial with real coefficients can be factored over the real numbers as a product of linear factors and irreducible quadratic factors. Once we have this factorization of the denominator of a rational function, the next theorem tells us the form the decomposition takes. The reader is encouraged to review the Factor Theorem (Theorem 3.6) and its connection to the role of multiplicity to fully appreciate the statement of the following theorem.

[3]We will justify this shortly.

Theorem 8.10. Suppose $R(x) = \dfrac{N(x)}{D(x)}$ is a rational function where the degree of $N(x)$ less than the degree of $D(x)$ [a] and $N(x)$ and $D(x)$ have no common factors.

- If c is a real zero of D of multiplicity m which corresponds to the linear factor $ax+b$, the partial fraction decomposition includes

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_m}{(ax+b)^m}$$

  for real numbers A_1, A_2, $\ldots A_m$.

- If c is a non-real zero of D of multiplicity m which corresponds to the irreducible quadratic ax^2+bx+c, the partial fraction decomposition includes

$$\frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \ldots + \frac{B_mx + C_m}{(ax^2+bx+c)^m}$$

  for real numbers B_1, B_2, $\ldots B_m$ and C_1, C_2, $\ldots C_m$.

[a] In other words, $R(x)$ is a proper rational function.

The proof of Theorem 8.10 is best left to a course in Abstract Algebra. Notice that the theorem provides for the general case, so we need to use subscripts, A_1, A_2, etc., to denote different unknown coefficients as opposed to the usual convention of A, B, etc.. The stress on multiplicities is to help us correctly group factors in the denominator. For example, consider the rational function

$$\frac{3x-1}{\left(x^2-1\right)\left(2-x-x^2\right)}$$

Factoring the denominator to find the zeros, we get $(x+1)(x-1)(1-x)(2+x)$. We find $x=-1$ and $x=-2$ are zeros of multiplicity one but that $x=1$ is a zero of multiplicity two due to the two different factors $(x-1)$ and $(1-x)$. One way to handle this is to note that $(1-x) = -(x-1)$ so

$$\frac{3x-1}{(x+1)(x-1)(1-x)(2+x)} = \frac{3x-1}{-(x-1)^2(x+1)(x+2)} = \frac{1-3x}{(x-1)^2(x+1)(x+2)}$$

from which we proceed with the partial fraction decomposition

$$\frac{1-3x}{(x-1)^2(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{x+2}$$

Turning our attention to non-real zeros, we note that the tool of choice to determine the irreducibility of a quadratic ax^2+bx+c is the discriminant, b^2-4ac. If $b^2-4ac<0$, the quadratic admits a *pair* of non-real complex conjugate zeros. Even though *one* irreducible quadratic gives *two* distinct non-real zeros, we list the terms with denominators involving a given irreducible quadratic only once to avoid duplication in the form of the decomposition. The trick, of course, is factoring the denominator or otherwise finding the zeros and their multiplicities in order to apply Theorem 8.10. We recommend that the reader review the techniques set forth in Sections 3.3 and 3.4. Next, we state a theorem that if two polynomials are equal, the corresponding coefficients of the like powers of x are equal. This is the principal by which we shall determine the unknown coefficients in our partial fraction decomposition.

Theorem 8.11. Suppose

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = b_mx^m + m_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0$$

for all x in an open interval I. Then $n = m$ and $a_i = b_i$ for all $i = 1 \ldots n$.

Believe it or not, the proof of Theorem 8.11 is a consequence of Theorem 3.14. Define $p(x)$ to be the difference of the left hand side of the equation in Theorem 8.11 and the right hand side. Then $p(x) = 0$ for all x in the open interval I. If $p(x)$ were a nonzero polynomial of degree k, then, by Theorem 3.14, p could have at most k zeros in I, and k is a finite number. Since $p(x) = 0$ for all the x in I, p has infinitely many zeros, and hence, p is the zero polynomial. This means there can be no nonzero terms in $p(x)$ and the theorem follows. Arguably, the best way to make sense of either of the two preceding theorems is to work some examples.

Example 8.6.1. Resolve the following rational functions into partial fractions.

1. $R(x) = \dfrac{x+5}{2x^2-x-1}$
2. $R(x) = \dfrac{3}{x^3-2x^2+x}$
3. $R(x) = \dfrac{3}{x^3-x^2+x}$
4. $R(x) = \dfrac{4x^3}{x^2-2}$
5. $R(x) = \dfrac{x^3+5x-1}{x^4+6x^2+9}$
6. $R(x) = \dfrac{8x^2}{x^4+16}$

Solution.

1. We begin by factoring the denominator to find $2x^2 - x - 1 = (2x + 1)(x - 1)$. We get $x = -\frac{1}{2}$ and $x = 1$ are both zeros of multiplicity one and thus we know

$$\frac{x+5}{2x^2-x-1} = \frac{x+5}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

Clearing denominators, we get $x+5 = A(x-1)+B(2x+1)$ so that $x+5 = (A+2B)x+B-A$. Equating coefficients, we get the system

$$\left\{ \begin{array}{rcl} A + 2B & = & 1 \\ -A + B & = & 5 \end{array} \right.$$

This system is readily handled using the Addition Method from Section 8.1, and after adding both equations, we get $3B = 6$ so $B = 2$. Using back substitution, we find $A = -3$. Our answer is easily checked by getting a common denominator and adding the fractions.

$$\frac{x+5}{2x^2-x-1} = \frac{2}{x-1} - \frac{3}{2x+1}$$

2. Factoring the denominator gives $x^3-2x^2+x = x\left(x^2-2x+1\right) = x(x-1)^2$ which gives $x=0$ as a zero of multiplicity one and $x=1$ as a zero of multiplicity two. We have

$$\frac{3}{x^3-2x^2+x} = \frac{3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Clearing denominators, we get $3 = A(x-1)^2 + Bx(x-1) + Cx$, which, after gathering up the like terms becomes $3 = (A+B)x^2 + (-2A-B+C)x + A$. Our system is

$$\left\{ \begin{array}{rcl} A+B & = & 0 \\ -2A-B+C & = & 0 \\ A & = & 3 \end{array} \right.$$

Substituting $A=3$ into $A+B=0$ gives $B=-3$, and substituting both for A and B in $-2A-B+C=0$ gives $C=3$. Our final answer is

$$\frac{3}{x^3-2x^2+x} = \frac{3}{x} - \frac{3}{x-1} + \frac{3}{(x-1)^2}$$

3. The denominator factors as $x\left(x^2-x+1\right)$. We see immediately that $x=0$ is a zero of multiplicity one, but the zeros of x^2-x+1 aren't as easy to discern. The quadratic doesn't factor easily, so we check the discriminant and find it to be $(-1)^2-4(1)(1) = -3 < 0$. We find its zeros are not real so it is an irreducible quadratic. The form of the partial fraction decomposition is then

$$\frac{3}{x^3-x^2+x} = \frac{3}{x\left(x^2-x+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2-x+1}$$

Proceeding as usual, we clear denominators and get $3 = A\left(x^2-x+1\right) + (Bx+C)x$ or $3 = (A+B)x^2 + (-A+C)x + A$. We get

$$\left\{ \begin{array}{rcl} A+B & = & 0 \\ -A+C & = & 0 \\ A & = & 3 \end{array} \right.$$

From $A=3$ and $A+B=0$, we get $B=-3$. From $-A+C=0$, we get $C=A=3$. We get

$$\frac{3}{x^3-x^2+x} = \frac{3}{x} + \frac{3-3x}{x^2-x+1}$$

4. Since $\frac{4x^3}{x^2-2}$ isn't proper, we use long division and we get a quotient of $4x$ with a remainder of $8x$. That is, $\frac{4x^3}{x^2-2} = 4x + \frac{8x}{x^2-2}$ so we now work on resolving $\frac{8x}{x^2-2}$ into partial fractions. The quadratic x^2-2, though it doesn't factor nicely, is, nevertheless, reducible. Solving $x^2-2=0$

gives us $x = \pm\sqrt{2}$, and each of these zeros must be of multiplicity one since Theorem 3.14 enables us to now factor $x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$. Hence,

$$\frac{8x}{x^2-2} = \frac{8x}{\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}}$$

Clearing fractions, we get $8x = A\left(x+\sqrt{2}\right) + B\left(x - \sqrt{2}\right)$ or $8x = (A+B)x + (A-B)\sqrt{2}$. We get the system

$$\left\{\begin{array}{rcl} A+B & = & 8 \\ (A-B)\sqrt{2} & = & 0 \end{array}\right.$$

From $(A-B)\sqrt{2} = 0$, we get $A = B$, which, when substituted into $A + B = 8$ gives $B = 4$. Hence, $A = B = 4$ and we get

$$\frac{4x^3}{x^2-2} = 4x + \frac{8x}{x^2-2} = 4x + \frac{4}{x+\sqrt{2}} + \frac{4}{x-\sqrt{2}}$$

5. At first glance, the denominator $D(x) = x^4 + 6x^2 + 9$ appears irreducible. However, $D(x)$ has three terms, and the exponent on the first term is exactly twice that of the second. Rewriting $D(x) = \left(x^2\right)^2 + 6x^2 + 9$, we see it is a quadratic in disguise and factor $D(x) = \left(x^2+3\right)^2$. Since $x^2 + 3$ clearly has no real zeros, it is irreducible and the form of the decomposition is

$$\frac{x^3+5x-1}{x^4+6x^2+9} = \frac{x^3+5x-1}{\left(x^2+3\right)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{\left(x^2+3\right)^2}$$

When we clear denominators, we find $x^3 + 5x - 1 = (Ax + B)\left(x^2 + 3\right) + Cx + D$ which yields $x^3 + 5x - 1 = Ax^3 + Bx^2 + (3A + C)x + 3B + D$. Our system is

$$\left\{\begin{array}{rcr} A & = & 1 \\ B & = & 0 \\ 3A+C & = & 5 \\ 3B+D & = & -1 \end{array}\right.$$

We have $A = 1$ and $B = 0$ from which we get $C = 2$ and $D = -1$. Our final answer is

$$\frac{x^3+5x-1}{x^4+6x^2+9} = \frac{x}{x^2+3} + \frac{2x-1}{\left(x^2+3\right)^2}$$

6. Once again, the difficulty in our last example is factoring the denominator. In an attempt to get a quadratic in disguise, we write

$$x^4 + 16 = \left(x^2\right)^2 + 4^2 = \left(x^2\right)^2 + 8x^2 + 4^2 - 8x^2 = \left(x^2+4\right)^2 - 8x^2$$

and obtain a difference of two squares: $\left(x^2+4\right)^2$ and $8x^2 = \left(2x\sqrt{2}\right)^2$. Hence,

$$x^4+16 = \left(x^2+4-2x\sqrt{2}\right)\left(x^2+4+2x\sqrt{2}\right) = \left(x^2-2x\sqrt{2}+4\right)\left(x^2+2x\sqrt{2}+4\right)$$

The discrimant of both of these quadratics works out to be $-8 < 0$, which means they are irreducible. We leave it to the reader to verify that, despite having the same discriminant, these quadratics have different zeros. The partial fraction decomposition takes the form

$$\frac{8x^2}{x^4+16} = \frac{8x^2}{\left(x^2-2x\sqrt{2}+4\right)\left(x^2+2x\sqrt{2}+4\right)} = \frac{Ax+B}{x^2-2x\sqrt{2}+4} + \frac{Cx+D}{x^2+2x\sqrt{2}+4}$$

We get $8x^2 = (Ax+B)\left(x^2+2x\sqrt{2}+4\right) + (Cx+D)\left(x^2-2x\sqrt{2}+4\right)$ or

$$8x^2 = (A+C)x^3 + (2A\sqrt{2}+B-2C\sqrt{2}+D)x^2 + (4A+2B\sqrt{2}+4C-2D\sqrt{2})x + 4B+4D$$

which gives the system

$$\left\{\begin{array}{rcl} A+C & = & 0 \\ 2A\sqrt{2}+B-2C\sqrt{2}+D & = & 8 \\ 4A+2B\sqrt{2}+4C-2D\sqrt{2} & = & 0 \\ 4B+4D & = & 0 \end{array}\right.$$

We choose substitution as the weapon of choice to solve this system. From $A+C=0$, we get $A=-C$; from $4B+4D=0$, we get $B=-D$. Substituting these into the remaining two equations, we get

$$\left\{\begin{array}{rcl} -2C\sqrt{2}-D-2C\sqrt{2}+D & = & 8 \\ -4C-2D\sqrt{2}+4C-2D\sqrt{2} & = & 0 \end{array}\right.$$

or

$$\left\{\begin{array}{rcl} -4C\sqrt{2} & = & 8 \\ -4D\sqrt{2} & = & 0 \end{array}\right.$$

We get $C=-\sqrt{2}$ so that $A=-C=\sqrt{2}$ and $D=0$ which means $B=-D=0$. We get

$$\frac{8x^2}{x^4+16} = \frac{x\sqrt{2}}{x^2-2x\sqrt{2}+4} - \frac{x\sqrt{2}}{x^2+2x\sqrt{2}+4}$$

□

8.6.1 Exercises

In Exercises 1 - 6, find only the *form* needed to begin the process of partial fraction decomposition. Do not create the system of linear equations or attempt to find the actual decomposition.

1. $\dfrac{7}{(x-3)(x+5)}$
2. $\dfrac{5x+4}{x(x-2)(2-x)}$
3. $\dfrac{m}{(7x-6)(x^2+9)}$
4. $\dfrac{ax^2+bx+c}{x^3(5x+9)(3x^2+7x+9)}$
5. $\dfrac{\text{A polynomial of degree } < 9}{(x+4)^5(x^2+1)^2}$
6. $\dfrac{\text{A polynomial of degree } < 7}{x(4x-1)^2(x^2+5)(9x^2+16)}$

In Exercises 7 - 18, find the partial fraction decomposition of the following rational expressions.

7. $\dfrac{2x}{x^2-1}$
8. $\dfrac{-7x+43}{3x^2+19x-14}$
9. $\dfrac{11x^2-5x-10}{5x^3-5x^2}$
10. $\dfrac{-2x^2+20x-68}{x^3+4x^2+4x+16}$
11. $\dfrac{-x^2+15}{4x^4+40x^2+36}$
12. $\dfrac{-21x^2+x-16}{3x^3+4x^2-3x+2}$
13. $\dfrac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2}$
14. $\dfrac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80}$
15. $\dfrac{-7x^2-76x-208}{x^3+18x^2+108x+216}$
16. $\dfrac{-10x^4+x^3-19x^2+x-10}{x^5+2x^3+x}$
17. $\dfrac{4x^3-9x^2+12x+12}{x^4-4x^3+8x^2-16x+16}$
18. $\dfrac{2x^2+3x+14}{(x^2+2x+9)(x^2+x+5)}$
19. As we stated at the beginning of this section, the technique of resolving a rational function into partial fractions is a skill needed for Calculus. However, we hope to have shown you that it is worth doing if, for no other reason, it reinforces a hefty amount of algebra. One of the common algebraic errors the authors find students make is something along the lines of

$$\frac{8}{x^2-9} \neq \frac{8}{x^2} - \frac{8}{9}$$

Think about why if the above were true, this section would have no need to exist.

8.6.2 Answers

1. $\frac{A}{x-3}+\frac{B}{x+5}$

2. $\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

3. $\frac{A}{7x-6}+\frac{Bx+C}{x^2+9}$

4. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{5x+9}+\frac{Ex+F}{3x^2+7x+9}$

5. $\frac{A}{x+4}+\frac{B}{(x+4)^2}+\frac{C}{(x+4)^3}+\frac{D}{(x+4)^4}+\frac{E}{(x+4)^5}+\frac{Fx+G}{x^2+1}+\frac{Hx+I}{(x^2+1)^2}$

6. $\frac{A}{x}+\frac{B}{4x-1}+\frac{C}{(4x-1)^2}+\frac{Dx+E}{x^2+5}+\frac{Fx+G}{9x^2+16}$

7. $\frac{2x}{x^2-1}=\frac{1}{x+1}+\frac{1}{x-1}$

8. $\frac{-7x+43}{3x^2+19x-14}=\frac{5}{3x-2}-\frac{4}{x+7}$

9. $\frac{11x^2-5x-10}{5x^3-5x^2}=\frac{3}{x}+\frac{2}{x^2}-\frac{4}{5(x-1)}$

10. $\frac{-2x^2+20x-68}{x^3+4x^2+4x+16}=-\frac{9}{x+4}+\frac{7x-8}{x^2+4}$

11. $\frac{-x^2+15}{4x^4+40x^2+36}=\frac{1}{2(x^2+1)}-\frac{3}{4(x^2+9)}$

12. $\frac{-21x^2+x-16}{3x^3+4x^2-3x+2}=-\frac{6}{x+2}-\frac{3x+5}{3x^2-2x+1}$

13. $\frac{5x^4-34x^3+70x^2-33x-19}{(x-3)^2}=5x^2-4x+1+\frac{9}{x-3}-\frac{1}{(x-3)^2}$

14. $\frac{x^6+5x^5+16x^4+80x^3-2x^2+6x-43}{x^3+5x^2+16x+80}=x^3+\frac{x+1}{x^2+16}-\frac{3}{x+5}$

15. $\frac{-7x^2-76x-208}{x^3+18x^2+108x+216}=-\frac{7}{x+6}+\frac{8}{(x+6)^2}-\frac{4}{(x+6)^3}$

16. $\frac{-10x^4+x^3-19x^2+x-10}{x^5+2x^3+x}=-\frac{10}{x}+\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}$

17. $\frac{4x^3-9x^2+12x+12}{x^4-4x^3+8x^2-16x+16}=\frac{1}{x-2}+\frac{4}{(x-2)^2}+\frac{3x+1}{x^2+4}$

18. $\frac{2x^2+3x+14}{(x^2+2x+9)(x^2+x+5)}=\frac{1}{x^2+2x+9}+\frac{1}{x^2+x+5}$

8.7 Systems of Non-Linear Equations and Inequalities

In this section, we study systems of non-linear equations and inequalities. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

Example 8.7.1. Solve the following systems of equations. Verify your answers algebraically and graphically.

1. $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 + 9y^2 & = & 36 \end{array} \right.$

2. $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 - 9y^2 & = & 36 \end{array} \right.$

3. $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ y - 2x & = & 0 \end{array} \right.$

4. $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ y - x^2 & = & 0 \end{array} \right.$

Solution:

1. Since both equations contain x^2 and y^2 only, we can eliminate one of the variables as we did in Section 8.1.

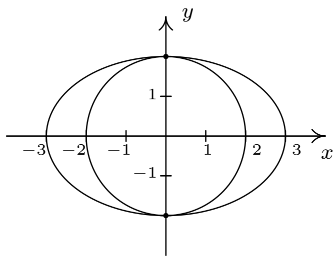
$$\left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 4x^2 + 9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 5y^2 & = & 20 \end{array} \right.$$

From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0, 2), (0, -2)\}$. To check this algebraically, we need to show that both points satisfy both of the original equations. We leave it to the reader to verify this. To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is easily recognized as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4. We see from the graph that the two curves intersect at their y-intercepts only, $(0, \pm 2)$.

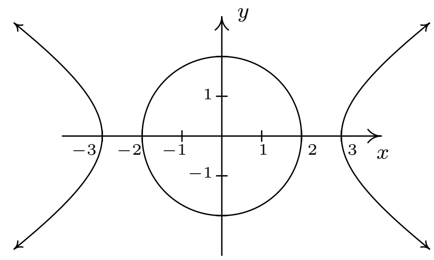
2. We proceed as before to eliminate one of the variables

$$\left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 4x^2 - 9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{array} \right.$$

Since the equation $-13y^2 = 20$ admits no real solution, the system is inconsistent. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0,0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. We see that the circle and the hyperbola have no points in common.



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 - 9y^2 = 36 \end{cases}$

3. Since there are no like terms among the two equations, elimination won't do us any good. We turn to substitution and from the equation $y - 2x = 0$, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Solving, we find $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$. Returning to the equation we used for the substitution, $y = 2x$, we find $y = \frac{4\sqrt{5}}{5}$ when $x = \frac{2\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$. Similarly, we find the other solution to be $\left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)$. We leave it to the reader that both points satisfy both equations, so that our final answer is $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)\right\}$. The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$ is a line through the origin with slope 2. Though we cannot verify the numerical values of the points of intersection from our sketch, we do see that we have two solutions: one in Quadrant I and one in Quadrant III as required.

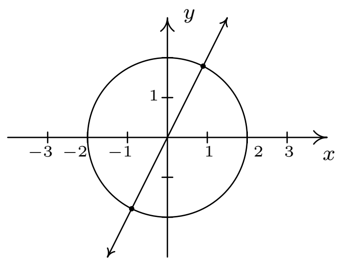
4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination.[1]

$$\begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & y - x^2 = 0 \end{cases} \xrightarrow[E1+E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & y^2 + y = 4 \end{cases}$$

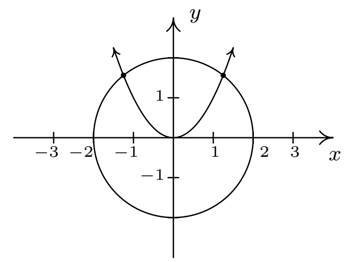
From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$ which gives $y = \frac{-1\pm\sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations to head off any extraneous solutions we may encounter. We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly twice, and both of these points have a positive y value. Of the two solutions for y, only $y = \frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this

[1] We encourage the reader to solve the system using substitution to see that you get the same solution.

into $y - x^2 = 0$ and solve for x. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$. Our solution is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$, which we leave to the reader to verify.



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ y - 2x = 0 \end{cases}$



Graphs for $\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 36 \end{cases}$

□

A couple of remarks about Example 8.7.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'. Secondly, as we saw with number 4, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it sometimes pays to take advantage when they are. Our next example provides some considerable review of many of the topics introduced in this text.

Example 8.7.2. Solve the following systems of equations. Verify your answers algebraically and graphically, as appropriate.

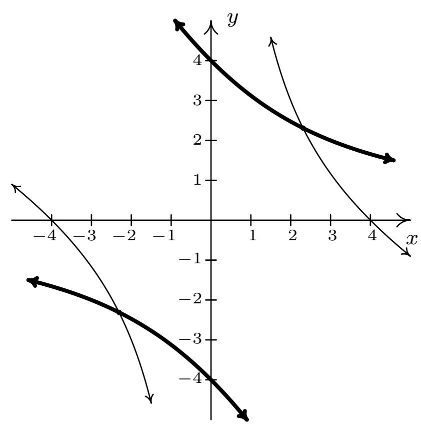
1. $\begin{cases} x^2 + 2xy - 16 = 0 \\ y^2 + 2xy - 16 = 0 \end{cases}$
2. $\begin{cases} y + 4e^{2x} = 1 \\ y^2 + 2e^x = 1 \end{cases}$
3. $\begin{cases} z(x-2) = x \\ yz = y \\ (x-2)^2 + y^2 = 1 \end{cases}$

Solution.

1. At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome xy term, and the constant term as well, by elimination and doing so we get a more tractable relationship between x and y

$$\begin{cases} (E1) & x^2 + 2xy - 16 = 0 \\ (E2) & y^2 + 2xy - 16 = 0 \end{cases} \xrightarrow[-E1+E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + 2xy - 16 = 0 \\ (E2) & y^2 - x^2 = 0 \end{cases}$$

We get $y^2 - x^2 = 0$ or $y = \pm x$. Substituting $y = x$ into $E1$ we get $x^2 + 2x^2 - 16 = 0$ so that $x^2 = \frac{16}{3}$ or $x = \pm\frac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into $E1$, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$ which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. We leave it to the reader to show that both points satisfy both equations and now turn to verifying our solution graphically. We begin by solving $x^2+2xy-16 = 0$ for y to obtain $y = \frac{16-x^2}{2x}$. This function is easily graphed using the techniques of Section 4.2. Solving the second equation, $y^2 + 2xy - 16 = 0$, for y, however, is more complicated. We use the quadratic formula to obtain $y = -x \pm \sqrt{x^2 + 16}$ which would require the use of Calculus or a calculator to graph. Believe it or not, we don't need either because the equation $y^2+2xy-16 = 0$ can be obtained from the equation $x^2 + 2xy - 16 = 0$ by interchanging y and x. Thinking back to Section 5.2, this means we can obtain the graph of $y^2 + 2xy - 16 = 0$ by reflecting the graph of $x^2 + 2xy - 16 = 0$ across the line $y = x$. Doing so confirms that the two graphs intersect twice: once in Quadrant I, and once in Quadrant III as required.



The graphs of $x^2 + 2xy - 16 = 0$ and $\boldsymbol{y^2 + 2xy - 16 = 0}$

2. Unlike the previous problem, there seems to be no avoiding substitution and a bit of algebraic unpleasantness. Solving $y + 4e^{2x} = 1$ for y, we get $y = 1 - 4e^{2x}$ which, when substituted into the second equation, yields $\left(1 - 4e^{2x}\right)^2 + 2e^x = 1$. After expanding and gathering like terms, we get $16e^{4x} - 8e^{2x} + 2e^x = 0$. Factoring gives us $2e^x\left(8e^{3x} - 4e^x + 1\right) = 0$, and since $2e^x \neq 0$ for any real x, we are left with solving $8e^{3x} - 4e^x + 1 = 0$. We have three terms, and even though this is not a 'quadratic in disguise', we can benefit from the substitution $u = e^x$. The equation becomes $8u^3-4u+1 = 0$. Using the techniques set forth in Section 3.3, we find $u = \frac{1}{2}$ is a zero and use synthetic division to factor the left hand side as $\left(u - \frac{1}{2}\right)\left(8u^2 + 4u - 2\right)$. We use the quadratic formula to solve $8u^2 + 4u - 2 = 0$ and find $u = \frac{-1\pm\sqrt{5}}{4}$. Since $u = e^x$, we now must solve $e^x = \frac{1}{2}$ and $e^x = \frac{-1\pm\sqrt{5}}{4}$. From $e^x = \frac{1}{2}$, we get $x = \ln\left(\frac{1}{2}\right) = -\ln(2)$. As for $e^x = \frac{-1\pm\sqrt{5}}{4}$, we first note that $\frac{-1-\sqrt{5}}{4} < 0$, so $e^x = \frac{-1-\sqrt{5}}{4}$ has no real solutions. We are

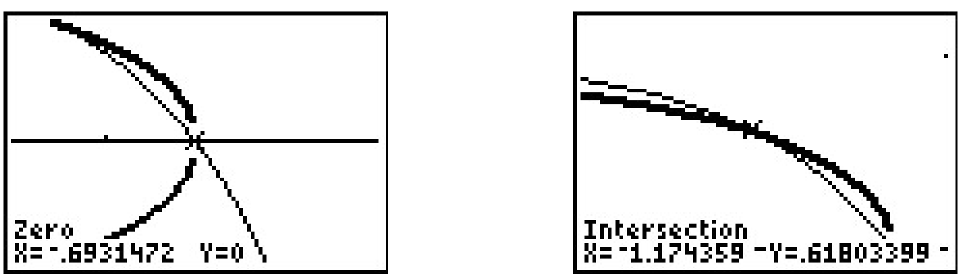
left with $e^x = \frac{-1+\sqrt{5}}{4}$, so that $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$. We now return to $y = 1 - 4e^{2x}$ to find the accompanying y values for each of our solutions for x. For $x = -\ln(2)$, we get

$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{-2\ln(2)} \\ &= 1 - 4e^{\ln\left(\frac{1}{4}\right)} \\ &= 1 - 4\left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

For $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$, we have

$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{2\ln\left(\frac{-1+\sqrt{5}}{4}\right)} \\ &= 1 - 4e^{\ln\left(\frac{-1+\sqrt{5}}{4}\right)^2} \\ &= 1 - 4\left(\frac{-1+\sqrt{5}}{4}\right)^2 \\ &= 1 - 4\left(\frac{3-\sqrt{5}}{8}\right) \\ &= \frac{-1+\sqrt{5}}{2} \end{aligned}$$

We get two solutions, $\left\{(0, -\ln(2)), \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)\right\}$. It is a good review of the properties of logarithms to verify both solutions, so we leave that to the reader. We are able to sketch $y = 1 - 4e^{2x}$ using transformations, but the second equation is more difficult and we resort to the calculator. We note that to graph $y^2 + 2e^x = 1$, we need to graph both the positive and negative roots, $y = \pm\sqrt{1 - 2e^x}$. After some careful zooming,[2] we get



The graphs of $y = 1 - 4e^{2x}$ and $\boldsymbol{y = \pm\sqrt{1 - 2e^x}}$.

3. Our last system involves three variables and gives some insight on how to keep such systems organized. Labeling the equations as before, we have

[2]The calculator has trouble confirming the solution $(-\ln(2), 0)$ due to its issues in graphing square root functions. If we mentally connect the two branches of the thicker curve, we see the intersection.

$$\begin{cases} E1 & z(x-2) & = & x \\ E2 & yz & = & y \\ E3 & (x-2)^2+y^2 & = & 1 \end{cases}$$

The easiest equation to start with appears to be $E2$. While it may be tempting to divide both sides of $E2$ by y, we caution against this practice because it presupposes $y \neq 0$. Instead, we take $E2$ and rewrite it as $yz - y = 0$ so $y(z-1) = 0$. From this, we get two cases: $y = 0$ or $z = 1$. We take each case in turn.

Case 1: $y = 0$. Substituting $y = 0$ into $E1$ and $E3$, we get

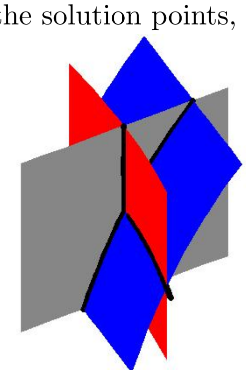
$$\begin{cases} E1 & z(x-2) & = & x \\ E3 & (x-2)^2 & = & 1 \end{cases}$$

Solving $E3$ for x gives $x = 1$ or $x = 3$. Substituting these values into $E1$ gives $z = -1$ when $x = 1$ and $z = 3$ when $x = 3$. We obtain two solutions, $(1, 0, -1)$ and $(3, 0, 3)$.

Case 2: $z = 1$. Substituting $z = 1$ into $E1$ and $E3$ gives us

$$\begin{cases} E1 & (1)(x-2) & = & x \\ E3 & (1-2)^2+y^2 & = & 1 \end{cases}$$

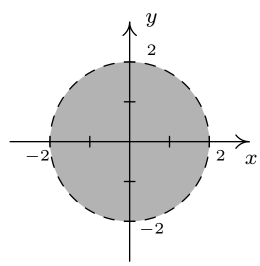
Equation $E1$ gives us $x - 2 = x$ or $-2 = 0$, which is a contradiction. This means we have no solution to the system in this case, even though $E3$ is solvable and gives $y = 0$. Hence, our final answer is $\{(1, 0, -1), (3, 0, 3)\}$. These points are easy enough to check algebraically in our three original equations, so that is left to the reader. As for verifying these solutions graphically, they require plotting surfaces in three dimensions and looking for intersection points. While this is beyond the scope of this book, we provide a snapshot of the graphs of our three equations near one of the solution points, $(1, 0, -1)$.



□

Example 8.7.2 showcases some of the ingenuity and tenacity mentioned at the beginning of the section. Sometimes you just have to look at a system the right way to find the most efficient method to solve it. Sometimes you just have to try something.

We close this section discussing how non-linear inequalities can be used to describe regions in the plane which we first introduced in Section 2.4. Before we embark on some examples, a little motivation is in order. Suppose we wish to solve $x^2 < 4 - y^2$. If we mimic the algorithms for solving nonlinear inequalities in one variable, we would gather all of the terms on one side and leave a 0 on the other to obtain $x^2 + y^2 - 4 < 0$. Then we would find the zeros of the left hand side, that is, where is $x^2 + y^2 - 4 = 0$, or $x^2 + y^2 = 4$. Instead of obtaining a few *numbers* which divide the real number *line* into *intervals*, we get an equation of a *curve*, in this case, a circle, which divides the *plane* into two *regions* - the 'inside' and 'outside' of the circle - with the circle itself as the boundary between the two. Just like we used test *values* to determine whether or not an interval belongs to the solution of the inequality, we use test *points* in the each of the regions to see which of these belong to our solution set.[3] We choose $(0, 0)$ to represent the region inside the circle and $(0, 3)$ to represent the points outside of the circle. When we substitute $(0, 0)$ into $x^2 + y^2 - 4 < 0$, we get $-4 < 4$ which is true. This means $(0, 0)$ and all the other points inside the circle are part of the solution. On the other hand, when we substitute $(0, 3)$ into the same inequality, we get $5 < 0$ which is false. This means $(0, 3)$ along with all other points outside the circle are not part of the solution. What about points on the circle itself? Choosing a point on the circle, say $(0, 2)$, we get $0 < 0$, which means the circle itself does not satisfy the inequality.[4] As a result, we leave the circle dashed in the final diagram.



The solution to $x^2 < 4 - y^2$

We put this technique to good use in the following example.

Example 8.7.3. Sketch the solution to the following nonlinear inequalities in the plane.

1. $y^2 - 4 \leq x < y + 2$
2. $\left\{ \begin{array}{rcl} x^2 + y^2 & \geq & 4 \\ x^2 - 2x + y^2 - 2y & \leq & 0 \end{array} \right.$

Solution.

1. The inequality $y^2 - 4 \leq x < y + 2$ is a compound inequality. It translates as $y^2 - 4 \leq x$ and $x < y + 2$. As usual, we solve each inequality and take the set theoretic intersection to determine the region which satisfies both inequalities. To solve $y^2 - 4 \leq x$, we write

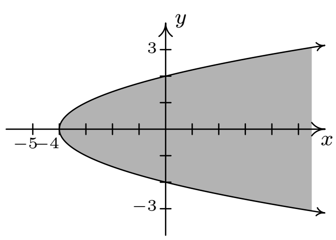
[3] The theory behind why all this works is, surprisingly, the same theory which guarantees that sign diagrams work the way they do - continuity and the Intermediate Value Theorem - but in this case, applied to functions of more than one variable.

[4] Another way to see this is that points on the circle satisfy $x^2 + y^2 - 4 = 0$, so they do not satisfy $x^2 + y^2 - 4 < 0$.

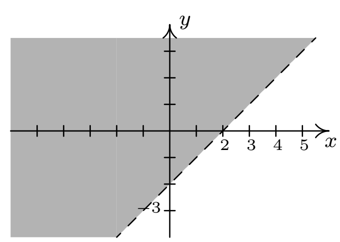
$y^2 - x - 4 \leq 0$. The curve $y^2 - x - 4 = 0$ describes a parabola since exactly one of the variables is squared. Rewriting this in standard form, we get $y^2 = x + 4$ and we see that the vertex is $(-4, 0)$ and the parabola opens to the right. Using the test points $(-5, 0)$ and $(0, 0)$, we find that the solution to the inequality includes the region to the right of, or 'inside', the parabola. The points on the parabola itself are also part of the solution, since the vertex $(-4, 0)$ satisfies the inequality. We now turn our attention to $x < y + 2$. Proceeding as before, we write $x - y - 2 < 0$ and focus our attention on $x - y - 2 = 0$, which is the line $y = x - 2$. Using the test points $(0, 0)$ and $(0, -4)$, we find points in the region above the line $y = x - 2$ satisfy the inequality. The points on the line $y = x - 2$ do not satisfy the inequality, since the y-intercept $(0, -2)$ does not. We see that these two regions do overlap, and to make the graph more precise, we seek the intersection of these two curves. That is, we need to solve the system of nonlinear equations

$$\begin{cases} (E1) & y^2 = x + 4 \\ (E2) & y = x - 2 \end{cases}$$

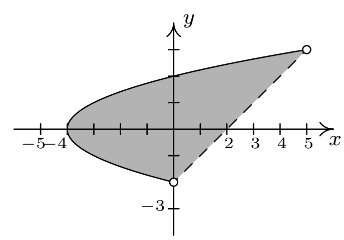
Solving $E1$ for x, we get $x = y^2 - 4$. Substituting this into $E2$ gives $y = y^2 - 4 - 2$, or $y^2 - y - 6 = 0$. We find $y = -2$ and $y = 3$ and since $x = y^2 - 4$, we get that the graphs intersect at $(0, -2)$ and $(5, 3)$. Putting all of this together, we get our final answer below.



$y^2 - 4 \leq x$



$x < y + 2$



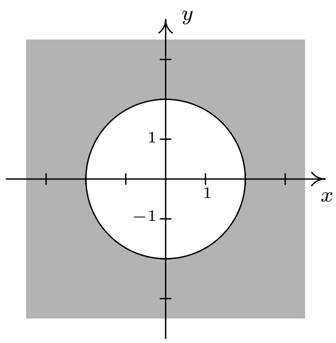
$y^2 - 4 \leq x < y + 2$

2. To solve this system of inequalities, we need to find all of the points (x, y) which satisfy both inequalities. To do this, we solve each inequality separately and take the set theoretic intersection of the solution sets. We begin with the inequality $x^2 + y^2 \geq 4$ which we rewrite as $x^2 + y^2 - 4 \geq 0$. The points which satisfy $x^2 + y^2 - 4 = 0$ form our friendly circle $x^2 + y^2 = 4$. Using test points $(0, 0)$ and $(0, 3)$ we find that our solution comprises the region outside the circle. As far as the circle itself, the point $(0, 2)$ satisfies the inequality, so the circle itself is part of the solution set. Moving to the inequality $x^2 - 2x + y^2 - 2y \leq 0$, we start with $x^2 - 2x + y^2 - 2y = 0$. Completing the squares, we obtain $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle centered at $(1, 1)$ with a radius of $\sqrt{2}$. Choosing $(1, 1)$ to represent the inside of the circle, $(1, 3)$ as a point outside of the circle and $(0, 0)$ as a point on the circle, we find that the solution to the inequality is the inside of the circle, including the circle itself. Our final answer, then, consists of the points on or outside of the circle $x^2 + y^2 = 4$ which lie on or

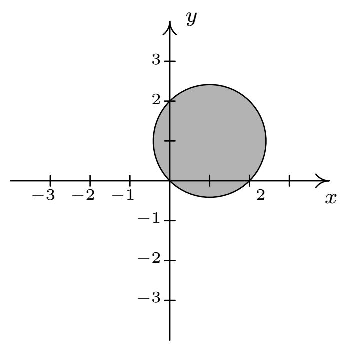
inside the circle $(x-1)^2 + (y-1)^2 = 2$. To produce the most accurate graph, we need to find where these circles intersect. To that end, we solve the system

$$\left\{ \begin{array}{rrcl} (E1) & x^2 + y^2 & = & 4 \\ (E2) & x^2 - 2x + y^2 - 2y & = & 0 \end{array} \right.$$

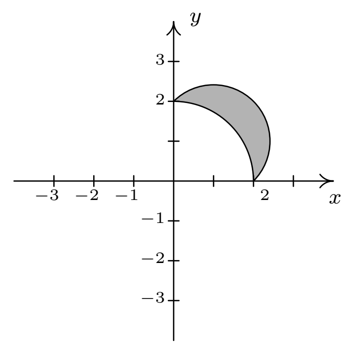
We can eliminate both the x^2 and y^2 by replacing $E2$ with $-E1 + E2$. Doing so produces $-2x - 2y = -4$. Solving this for y, we get $y = 2 - x$. Substituting this into $E1$ gives $x^2 + (2-x)^2 = 4$ which simplifies to $x^2 + 4 - 4x + x^2 = 4$ or $2x^2 - 4x = 0$. Factoring yields $2x(x-2)$ which gives $x = 0$ or $x = 2$. Substituting these values into $y = 2 - x$ gives the points $(0, 2)$ and $(2, 0)$. The intermediate graphs and final solution are below.



$x^2 + y^2 \geq 4$



$x^2 - 2x + y^2 - 2y \leq 0$



Solution to the system.

□

8.7.1 Exercises

In Exercises 1 - 6, solve the given system of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

1. $\left\{\begin{array}{rcl} x^2 - y & = & 4 \\ x^2 + y^2 & = & 4 \end{array}\right.$

2. $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ x^2 - y & = & 5 \end{array}\right.$

3. $\left\{\begin{array}{rcl} x^2 + y^2 & = & 16 \\ 16x^2 + 4y^2 & = & 64 \end{array}\right.$

4. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 16 \\ 9x^2 - 16y^2 & = & 144 \end{array}\right.$

5. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 16 \\ \frac{1}{9}y^2 - \frac{1}{16}x^2 & = & 1 \end{array}\right.$

6. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 16 \\ x - y & = & 2 \end{array}\right.$

In Exercises 9 - 15, solve the given system of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

7. $\left\{\begin{array}{rcl} x^2 - y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

8. $\left\{\begin{array}{rcl} \sqrt{x+1} - y & = & 0 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

9. $\left\{\begin{array}{rcl} x + 2y^2 & = & 2 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

10. $\left\{\begin{array}{rcl} (x-2)^2 + y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

11. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 25 \\ y - x & = & 1 \end{array}\right.$

12. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 25 \\ x^2 + (y-3)^2 & = & 10 \end{array}\right.$

13. $\left\{\begin{array}{rcr} y & = & x^3 + 8 \\ y & = & 10x - x^2 \end{array}\right.$

14. $\left\{\begin{array}{rcl} x^2 - xy & = & 8 \\ y^2 - xy & = & 8 \end{array}\right.$

15. $\left\{\begin{array}{rcr} x^2 + y^2 & = & 25 \\ 4x^2 - 9y & = & 0 \\ 3y^2 - 16x & = & 0 \end{array}\right.$

16. A certain bacteria culture follows the Law of Uninbited Growth, Equation 6.4. After 10 minutes, there are 10,000 bacteria. Five minutes later, there are 14,000 bacteria. How many bacteria were present initially? How long before there are 50,000 bacteria?

Consider the system of nonlinear equations below

$$\left\{\begin{array}{rcr} \dfrac{4}{x} + \dfrac{3}{y} & = & 1 \\ \dfrac{3}{x} + \dfrac{2}{y} & = & -1 \end{array}\right.$$

If we let $u = \frac{1}{x}$ and $v = \frac{1}{y}$ then the system becomes

$$\left\{\begin{array}{rcr} 4u + 3v & = & 1 \\ 3u + 2v & = & -1 \end{array}\right.$$

This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u = -5$ and $v = 7$. Thus $x = \frac{1}{u} = -\frac{1}{5}$ and $y = \frac{1}{v} = \frac{1}{7}$.

We say that the original system is **linear in form** because its equations are not linear but a few substitutions reveal a structure that we can treat like a system of linear equations. Each system in Exercises 17 - 19 is linear in form. Make the appropriate substitutions and solve for x and y.

17. $\begin{cases} 4x^3 + 3\sqrt{y} &= 1 \\ 3x^3 + 2\sqrt{y} &= -1 \end{cases}$

18. $\begin{cases} 4e^x + 3e^{-y} &= 1 \\ 3e^x + 2e^{-y} &= -1 \end{cases}$

19. $\begin{cases} 4\ln(x) + 3y^2 &= 1 \\ 3\ln(x) + 2y^2 &= -1 \end{cases}$

20. Solve the following system

$$\begin{cases} x^2 + \sqrt{y} + \log_2(z) &= 6 \\ 3x^2 - 2\sqrt{y} + 2\log_2(z) &= 5 \\ -5x^2 + 3\sqrt{y} + 4\log_2(z) &= 13 \end{cases}$$

In Exercises 21 - 26, sketch the solution to each system of nonlinear inequalities in the plane.

21. $\begin{cases} x^2 - y^2 &\leq 1 \\ x^2 + 4y^2 &\geq 4 \end{cases}$

22. $\begin{cases} x^2 + y^2 &< 25 \\ x^2 + (y-3)^2 &\geq 10 \end{cases}$

23. $\begin{cases} (x-2)^2 + y^2 &< 1 \\ x^2 + 4y^2 &< 4 \end{cases}$

24. $\begin{cases} y &> 10x - x^2 \\ y &< x^3 + 8 \end{cases}$

25. $\begin{cases} x + 2y^2 &> 2 \\ x^2 + 4y^2 &\leq 4 \end{cases}$

26. $\begin{cases} x^2 + y^2 &\geq 25 \\ y - x &\leq 1 \end{cases}$

27. Systems of nonlinear equations show up in third semester Calculus in the midst of some really cool problems. The system below came from a problem in which we were asked to find the dimensions of a rectangular box with a volume of 1000 cubic inches that has minimal surface area. The variables x, y and z are the dimensions of the box and λ is called a Lagrange multiplier. With the help of your classmates, solve the system.[5]

$$\begin{cases} 2y + 2z &= \lambda yz \\ 2x + 2z &= \lambda xz \\ 2y + 2x &= \lambda xy \\ xyz &= 1000 \end{cases}$$

28. According to Theorem 3.16 in Section 3.4, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

    (a) Show that p has no real zeros.

    (b) Because p has no real zeros, its factorization must be of the form $(x^2+ax+b)(x^2+cx+d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

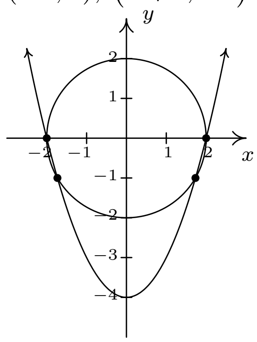
    (c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns a, b, c and d. Write $p(x)$ in factored form.

29. Factor $q(x) = x^4 + 6x^2 - 5x + 6$.

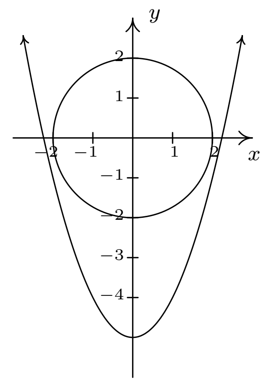
[5]If using λ bothers you, change it to w when you solve the system.

8.7.2 Answers

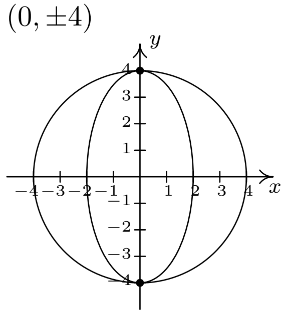
1. $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$



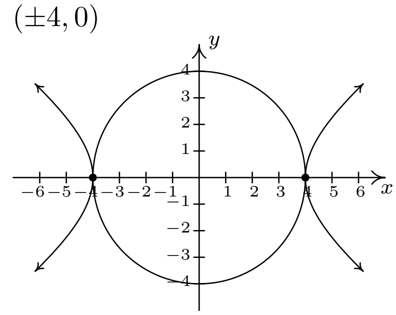
2. No solution



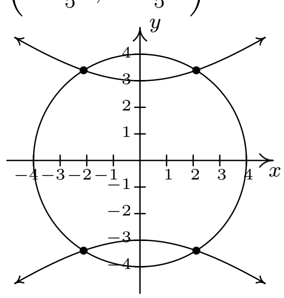
3. $(0, \pm 4)$



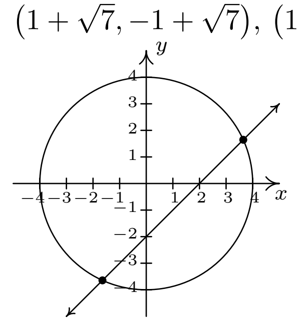
4. $(\pm 4, 0)$



5. $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$



6. $\left(1+\sqrt{7}, -1+\sqrt{7}\right)$, $\left(1-\sqrt{7}, -1-\sqrt{7}\right)$



7. $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$

8. $(0, 1)$

9. $(0, \pm 1)$, $(2, 0)$

10. $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{3}\right)$

11. $(3, 4)$, $(-4, -3)$

12. $(\pm 3, 4)$

13. $(-4, -56)$, $(1, 9)$, $(2, 16)$

14. $(-2, 2)$, $(2, -2)$

15. $(3, 4)$

16. Initially, there are $\frac{250000}{49} \approx 5102$ bacteria. It will take $\frac{5\ln(49/5)}{\ln(7/5)} \approx 33.92$ minutes for the colony to grow to 50,000 bacteria.

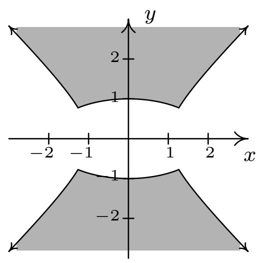
17. $\left(-\sqrt[3]{5}, 49\right)$

18. No solution

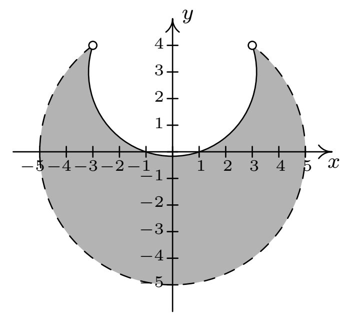
19. $\left(e^{-5}, \pm\sqrt{7}\right)$

20. $(1, 4, 8),\ (-1, 4, 8)$

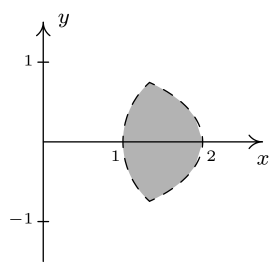
21. $\begin{cases} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{cases}$



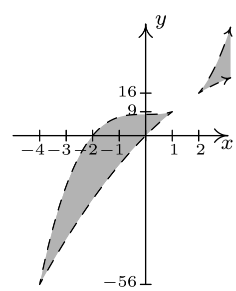
22. $\begin{cases} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{cases}$



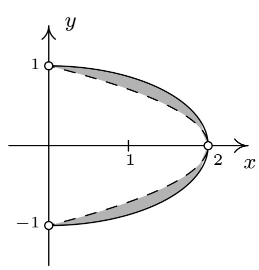
23. $\begin{cases} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{cases}$



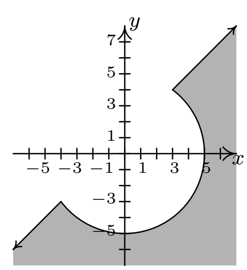
24. $\begin{cases} y & > & 10x - x^2 \\ y & < & x^3 + 8 \end{cases}$



25. $\begin{cases} x + 2y^2 & > & 2 \\ x^2 + 4y^2 & \leq & 4 \end{cases}$



26. $\begin{cases} x^2 + y^2 & \geq & 25 \\ y - x & \leq & 1 \end{cases}$



27. $x = 10,\ y = 10,\ z = 10, \lambda = \frac{2}{5}$

28. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

29. $x^4 + 6x^2 - 5x + 6 = (x^2 - x + 1)(x^2 + x + 6)$

Chapter 9

Sequences and the Binomial Theorem

9.1 Sequences

When we first introduced a function as a special type of relation in Section 1.3, we did not put any restrictions on the domain of the function. All we said was that the set of x-coordinates of the points in the function F is called the domain, and it turns out that any subset of the real numbers, regardless of how weird that subset may be, can be the domain of a function. As our exploration of functions continued beyond Section 1.3, we saw fewer and fewer functions with 'weird' domains. It is worth your time to go back through the text to see that the domains of the polynomial, rational, exponential, logarithmic and algebraic functions discussed thus far have fairly predictable domains which almost always consist of just a collection of intervals on the real line. This may lead some readers to believe that the only important functions in a College Algebra text have domains which consist of intervals and everything else was just introductory nonsense. In this section, we introduce **sequences** which are an important class of functions whose domains are the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \qquad (1)$$

As usual, the periods of ellipsis, $\ldots$, indicate that the proposed pattern continues forever. Each of the numbers in the list is called a **term**, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call a per tradition, between the natural numbers and the terms in the sequence.

[1] Recall that this is the set $\{1, 2, 3, \ldots\}$.

n	$a(n)$
1	$\frac{1}{2}$
2	$-\frac{3}{4}$
3	$\frac{9}{8}$
4	$-\frac{27}{16}$
$\vdots$	$\vdots$

In other words, $a(n)$ is the n^{th} term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

Definition 9.1. A **sequence** is a function a whose domain is the natural numbers. The value $a(n)$ is often written as a_n and is called the $\boldsymbol{n}^{\textbf{th}}$ **term** of the sequence. The sequence itself is usually denoted using the notation: a_n, $n \geq 1$ or the notation: $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 9.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$ and so forth. Now suppose we wanted to know a_{117}, that is, the 117^{th} term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for a_n. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

Example 9.1.1. Write the first four terms of the following sequences.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\left\{\dfrac{1+(-1)^i}{i}\right\}_{i=2}^{\infty}$
5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$
6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

Solution.

1. Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4. Since the notation a_1 means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2. For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3. Proceeding as before, replacing in this case the variable k with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. (This sequence is called an **alternating** sequence since the signs alternate between $+$ and $-$. The reader is encouraged to think what component of the formula is producing this effect.)

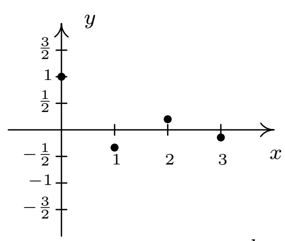
3. From $\{2n-1\}_{n=1}^{\infty}$, we have that $a_n = 2n-1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. (The first four terms are the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.)

4. Here, we are using the letter i as a counter, not as the imaginary unit we saw in Section 3.4. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. □

Some remarks about Example 9.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 9.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of k in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers. If you feel a sense of nostalgia, you should see Section 1.2.



Graphing $y = b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 9.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 9.1, we need to shift the variable k so it starts at $k = 1$ instead of $k = 0$. To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section 1.7, we can accomplish this by replacing k with $k-1$ in the definition of $\{b_k\}_{k=0}^{\infty}$. Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition

9.1, while the latter does not. Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 9.1 to include any function which generates a list of numbers which can then be matched up with the natural numbers.[2] Finally, we wish to note the sequences in parts 5 and 6 are examples of sequences described **recursively**. In each instance, an initial value of the sequence is given which is then followed by a **recursion equation** – a formula which enables us to use known terms of the sequence to determine other terms. The terms of the sequence in part 6 are given a special name: $f_n = n!$ is called ***n*-factorial**. Using the '!' notation, we can describe the factorial sequence as: $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. After $0! = 1$ the next four terms, written out in detail, are $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. From this, we see a more informal way of computing $n!$, which is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as a special case. (We will study factorials in greater detail in Section 9.4.) The world famous Fibonacci Numbers are defined recursively and are explored in the exercises. While none of the sequences worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

Definition 9.2. Arithmetic and Geometric Sequences: Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number d so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number d is called the **common difference**.
- If there is a number r so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **geometric sequence**. The number r is called the **common ratio**.

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

Both arithmetic and geometric sequences are defined in terms of recursion equations. In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number d. The name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number r. If $r \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$, hence the name 'common ratio.' Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[3]

Example 9.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d; if geometric, find the common ratio r.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

[2]We're basically talking about the 'countably infinite' subsets of the real number line when we do this.
[3]Sequences which are both arithmetic and geometric are discussed in the Exercises.

Solution. A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 9.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic. To investigate whether or not it is geometric, we compute the ratios of successive terms. The first three ratios
$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$
suggest that the sequence is geometric. To prove it, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n.
$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{(n+1)-1}}{3^{n+1}}}{\frac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$
This sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 9.1.1 to thank for providing the first four terms of this sequence: 1, $-\frac{1}{3}$, $\frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 9.1.1, the sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To verify this, we find
$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$
This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we assume that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

Equation 9.1. Formulas for Arithmetic and Geometric Sequences:

- An arithmetic sequence with first term a and common difference d is given by

$$a_n = a + (n-1)d, \quad n \geq 1$$

- A geometric sequence with first term a and common ratio $r \neq 0$ is given by

$$a_n = ar^{n-1}, \quad n \geq 1$$

While the formal proofs of the formulas in Equation 9.1 require the techniques set forth in Section 9.3, we attempt to motivate them here. According to Definition 9.2, given an arithmetic sequence with first term a and common difference d, the way we get from one term to the next is by adding d. Hence, the terms of the sequence are: a, $a+d$, $a+2d$, $a+3d$, $\ldots$. We see that to reach the nth term, we add d to a exactly $(n-1)$ times, which is what the formula says. The derivation of the formula for geometric series follows similarly. Here, we start with a and go from one term to the next by multiplying by r. We get a, ar, ar^2, ar^3 and so forth. The nth term results from multiplying a by r exactly $(n-1)$ times. We note here that the reason $r = 0$ is excluded from Equation 9.1 is to avoid an instance of 0^0 which is an indeterminant form.[4] With Equation 9.1 in place, we finally have the tools required to find an explicit formula for the nth term of the sequence given in (1). We know from Example 9.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 9.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of n and seeing that the formula generates the sequence given in (1). We leave this to the reader. Our next example gives us more practice finding patterns.

Example 9.1.3. Find an explicit formula for the n^{th} term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$
2. $\dfrac{2}{5}, 2, -\dfrac{2}{3}, -\dfrac{2}{7}, \ldots$
3. $1, -\dfrac{2}{7}, \dfrac{4}{13}, -\dfrac{8}{19}, \ldots$

Solution.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions. We have $\frac{2}{5}, \frac{2}{1}, -\frac{2}{3}, -\frac{2}{7}, \ldots$. If we associate the negative '$-$' of the last two terms with the denominators we get $\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, \frac{2}{-7}, \ldots$. This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{d_n}$ where d_n is the sequence of denominators.

[4]See the footnotes on page 237 in Section 3.1 and page 418 of Section 6.1.

Looking at the denominators $5, 1, -3, -7, \ldots$, we find that they go from one term to the next by subtracting 4 which is the same as adding -4. This means we have an arithmetic sequence on our hands. Using Equation 9.1 with $a = 5$ and $d = -4$, we get the nth denominator by the formula $d_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting c_n and d_n denote the sequence of numerators and denominators, respectively, we have $a_n = \frac{c_n}{d_n}$. After some experimentation,[5] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields $\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$. The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $c_n = (-2)^{n-1}$, for $n \geq 1$. The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $d_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$. We obtain our formula for $a_n = \frac{c_n}{d_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. □

While the last problem in Example 9.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for a_n if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the nth term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns. The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1+1, 4+1, 9+1, 16+1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$. Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[6] Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

Sequences play a major role in the Mathematics of Finance, as we have already seen with Equation 6.2 in Section 6.5. Recall that if we invest P dollars at an annual percentage rate r and compound the interest n times per year, the formula for A_k, the amount in the account after k compounding periods, is $A_k = P\left(1 + \frac{r}{n}\right)^k = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right)^{k-1}$, $k \geq 1$. We now spot this as a geometric sequence with first term $P\left(1 + \frac{r}{n}\right)$ and common ratio $\left(1 + \frac{r}{n}\right)$. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an **annuity** and will be discussed in the next section once we have developed more mathematical machinery.

[5] Here we take 'experimentation' to mean a frustrating guess-and-check session.

[6] For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test.

9.1.1 EXERCISES

In Exercises 1 - 13, write out the first four terms of the given sequence.

1. $a_n = 2^n - 1,\ n \geq 0$
2. $d_j = (-1)^{\frac{j(j+1)}{2}},\ j \geq 1$
3. $\{5k - 2\}_{k=1}^{\infty}$
4. $\left\{\dfrac{n^2+1}{n+1}\right\}_{n=0}^{\infty}$
5. $\left\{\dfrac{x^n}{n^2}\right\}_{n=1}^{\infty}$
6. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$
7. $a_1 = 3,\ a_{n+1} = a_n - 1,\ n \geq 1$
8. $d_0 = 12,\ d_m = \dfrac{d_{m-1}}{100},\ m \geq 1$
9. $b_1 = 2,\ b_{k+1} = 3b_k + 1,\ k \geq 1$
10. $c_0 = -2,\ c_j = \dfrac{c_{j-1}}{(j+1)(j+2)},\ m \geq 1$
11. $a_1 = 117,\ a_{n+1} = \dfrac{1}{a_n},\ n \geq 1$
12. $s_0 = 1,\ s_{n+1} = x^{n+1} + s_n,\ n \geq 0$
13. $F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ n \geq 2$ (This is the famous Fibonacci Sequence)

In Exercises 14 - 21 determine if the given sequence is arithmetic, geometric or neither. If it is arithmetic, find the common difference d; if it is geometric, find the common ratio r.

14. $\{3n - 5\}_{n=1}^{\infty}$
15. $a_n = n^2 + 3n + 2,\ n \geq 1$
16. $\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{12}, \dfrac{1}{24}, \ldots$
17. $\left\{3\left(\dfrac{1}{5}\right)^{n-1}\right\}_{n=1}^{\infty}$
18. $17, 5, -7, -19, \ldots$
19. $2, 22, 222, 2222, \ldots$
20. $0.9, 9, 90, 900, \ldots$
21. $a_n = \dfrac{n!}{2},\ n \geq 0.$

In Exercises 22 - 30, find an explicit formula for the n^{th} term of the given sequence. Use the formulas in Equation 9.1 as needed.

22. $3, 5, 7, 9, \ldots$
23. $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \ldots$
24. $1, \dfrac{2}{3}, \dfrac{4}{5}, \dfrac{8}{7}, \ldots$
25. $1, \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{4}{27}, \ldots$
26. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$
27. $x, -\dfrac{x^3}{3}, \dfrac{x^5}{5}, -\dfrac{x^7}{7}, \ldots$

28. $0.9, 0.99, 0.999, 0.9999, \ldots$

29. $27, 64, 125, 216, \ldots$

30. $1, 0, 1, 0, \ldots$

31. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all n.)

32. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.

33. Thomas Robert Malthus is credited with saying, "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second." (See this webpage for more information.) Discuss this quote with your classmates from a sequences point of view.

34. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the 30^{th} day? What is the total dollar value of the gift you have received?

35. Research the terms 'arithmetic mean' and 'geometric mean.' With the help of your classmates, show that a given term of a arithmetic sequence a_k, $k \geq 2$ is the arithmetic mean of the term immediately preceding, a_{k-1} it and immediately following it, a_{k+1}. State and prove an analogous result for geometric sequences.

36. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the n^{th} term of the sequence $i, -1, -i, 1, i, \ldots$. List out the first four terms of the matrix sequences we discussed in Exercise 8.3.1 in Section 8.3.

9.1.2 Answers

1. $0, 1, 3, 7$
2. $-1, -1, 1, 1$
3. $3, 8, 13, 18$
4. $1, 1, \frac{5}{3}, \frac{5}{2}$
5. $x, \frac{x^2}{4}, \frac{x^3}{9}, \frac{x^4}{16}$
6. $0, \frac{\ln(2)}{2}, \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$
7. $3, 2, 1, 0$
8. $12, 0.12, 0.0012, 0.000012$
9. $2, 7, 22, 67$
10. $-2, -\frac{1}{3}, -\frac{1}{36}, -\frac{1}{720}$
11. $117, \frac{1}{117}, 117, \frac{1}{117}$
12. $1, x+1, x^2+x+1, x^3+x^2+x+1$
13. $1, 1, 2, 3$
14. arithmetic, $d = 3$
15. neither
16. geometric, $r = \frac{1}{2}$
17. geometric, $r = \frac{1}{5}$
18. arithmetic, $d = -12$
19. neither
20. geometric, $r = 10$
21. neither
22. $a_n = 1 + 2n,\ n \geq 1$
23. $a_n = \left(-\frac{1}{2}\right)^{n-1},\ n \geq 1$
24. $a_n = \frac{2^{n-1}}{2n-1},\ n \geq 1$
25. $a_n = \frac{n}{3^{n-1}},\ n \geq 1$
26. $a_n = \frac{1}{n^2},\ n \geq 1$
27. $\frac{(-1)^{n-1}x^{2n-1}}{2n-1},\ n \geq 1$
28. $a_n = \frac{10^n-1}{10^n},\ n \geq 1$
29. $a_n = (n+2)^3,\ n \geq 1$
30. $a_n = \frac{1+(-1)^{n-1}}{2},\ n \geq 1$

9.2 Summation Notation

In the previous section, we introduced sequences and now we shall present notation and theorems concerning the sum of terms of a sequence. We begin with a definition, which, while intimidating, is meant to make our lives easier.

Definition 9.3. Summation Notation: Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers m and p satisfying $k \leq m \leq p$, the summation from m to p of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable n is called the **index of summation**. The number m is called the **lower limit of summation** while the number p is called the **upper limit of summation**.

In English, Definition 9.3 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from a_m through a_p. The symbol Σ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write the sum $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned} \sum_{n=3}^{6}(2n-1) &= (2(3)-1)+(2(4)-1)+(2(5)-1)+(2(6)-1) \\ &= 5+7+9+11 \\ &= 32 \end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers a_k, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition 3.1. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the ith row R_i of a matrix $A = [a_{ij}]_{m\times n}$ and the j^{th} column C_j of a matrix $B = [b_{ij}]_{n\times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition 8.9. Our next example gives us practice with this new notation.

Example 9.2.1.

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$ (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1 + 3 + 5 + \ldots + 117$

   (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117}$

   (c) $0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}}9$

Solution.

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned} \sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313 \end{aligned}$$

   (b) Proceeding as in (a), we replace every occurrence of n with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example 9.1.1, number 6 and get:

$$\begin{aligned} \sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{2 \cdot 1}{2} + \frac{3 \cdot 2 \cdot 1}{2} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2} \\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\ &= 17 \end{aligned}$$

   (c) We proceed as before, replacing the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 9.1.

   (a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. We get a formula for the nth term of the sequence using Equation 9.1 to get $a_n = 1 + (n-1)2 = 2n - 1$, $n \geq 1$. At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

   $$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59} (2n-1)$$

   (b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

   $$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

   The numerators, 1, -1, etc. can be described by the geometric sequence[1] $c_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $d_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

   (c) Thanks to Example 9.1.3, we know that one formula for the n^{th} term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation. To determine the upper limit of summation, we note that to produce the $n - 1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n. Hence, n is

[1]This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.

[2]It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.

the upper limit of summation. Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose k and get

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 \quad = \quad \sum_{k=1}^{n} \frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation. While we shall not have much need of these properties in Algebra, they do play a great role in Calculus. Moreover, there is much to be learned by thinking about why the properties hold. We invite the reader to prove these results. To get started, remember, "When in doubt, write it out!"

Theorem 9.1. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- $\displaystyle\sum_{n=m}^{p} (a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- $\displaystyle\sum_{n=m}^{p} c\, a_n = c \sum_{n=m}^{p} a_n$, for any real number c.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$, for any natural number $m \leq j < j+1 \leq p$.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any whole number r.

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S, we write it out in two different ways

$$\begin{array}{ccccccccccc} S & = & a & + & (a+d) & + & \ldots & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & \ldots & + & (a+d) & + & a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a+(n-1)d) + (2a+(n-1)d) + \ldots + (2a+(n-1)d) + (2a+(n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a + (n-1)d)$ so we get $2S = n(2a + (n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

[3] To see why, try writing the summation using 'n' as the index.

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS, we get

$$\begin{array}{rclllllllllllll} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & & \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a\left(1 - r^n\right)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1 - r^n}{1 - r}\right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1 - r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.

Equation 9.2. Sums of Arithmetic and Geometric Sequences:

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \sum_{k=1}^{n} a_k = \frac{a_1 - a_{n+1}}{1-r} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$.
2. $S = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 9.2, formal proofs require the machinery in Section 9.3. An application of the arithmetic sum formula which proves useful in Calculus results in formula for the sum of the first n natural numbers. The natural numbers themselves are a sequence[4] 1, 2, 3, ... which is arithmetic with $a = d = 1$. Applying Equation 9.2,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

So, for example, the sum of the first 100 natural numbers[5] is $\frac{100(101)}{2} = 5050$.

An important application of the geometric sum formula is the investment plan called an **annuity**. Annuities differ from the kind of investments we studied in Section 6.5 in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6] Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1(1+i) = P(1+i)$ in accordance with Equation 6.1. When we add our second payment at the end of the second period, we get

$$A_2 = A_1(1+i) + P = P(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)$$

The reason for factoring out the $P(1+i)$ will become apparent in short order. During the third compounding period, we earn interest on A_2 which then grows to $A_2(1+i)$. We add our third

[4]This is the identity function on the natural numbers!

[5]There is an interesting anecdote which says that the famous mathematician Carl Friedrich Gauss was given this problem in primary school and devised a very clever solution.

[6]The reader may wish to re-read the discussion on compound interest in Section 6.5 before proceeding.

payment at the end of the third compounding period to obtain

$$A_3 = A_2(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)(1+i) + P = P(1+i)^2\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the kth compounding, we get

$$A_k = P(1+i)^{k-1}\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 9.2, we get

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}} = 1\left(\frac{1 - \dfrac{1}{(1+i)^k}}{1 - \dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k - 1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an **ordinary annuity**.

Equation 9.3. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 9.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an **annuity-due**. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

Example 9.2.2. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.
2. How many years will it take for the annuity to grow to \$100,000?

Solution.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1 + 0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{aligned} 100000 &= \frac{50\left((1 + 0.005)^{12t} - 1\right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln\left((1.005)^{12t}\right) &= \ln(11) \\ 12t\ln(1.005) &= \ln(11) \\ t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{aligned}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. □

We close this section with a peek into Calculus by considering *infinite* sums, called **series**. Consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$

From Example 9.2.1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\cdots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + \ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n} \frac{9}{10^k}$$

Using Equation 9.2, we have

$$\sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10}\left(\frac{1-\dfrac{1}{10^{n+1}}}{1-\dfrac{1}{10}}\right) = 1 - \frac{1}{10^{n+1}}$$

It stands to reason that $0.\overline{9}$ is the same value of $1 - \frac{1}{10^{n+1}}$ as $n \to \infty$. Our knowledge of exponential expressions from Section 6.1 tells us that $\frac{1}{10^{n+1}} \to 0$ as $n \to \infty$, so $1 - \frac{1}{10^{n+1}} \to 1$. We have just argued that $0.\overline{9} = 1$, which may cause some distress for some readers.[7] Any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We end this section with a theorem concerning geometric series.

Theorem 9.2. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + \ldots = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + \ldots$ is not defined.

The justification of the result in Theorem 9.2 comes from taking the formula in Equation 9.2 for the sum of the first n terms of a geometric sequence and examining the formula as $n \to \infty$. Assuming $|r| < 1$ means $-1 < r < 1$, so $r^n \to 0$ as $n \to \infty$. Hence as $n \to \infty$,

$$\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right) \to \frac{a}{1-r}$$

As to what goes wrong when $|r| \geq 1$, we leave that to Calculus as well, but will explore some cases in the exercises.

[7] To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$. Feel better?

9.2.1 Exercises

In Exercises 1 - 8, find the value of each sum using Definition 9.3.

1. $\displaystyle\sum_{g=4}^{9}(5g+3)$
2. $\displaystyle\sum_{k=3}^{8}\frac{1}{k}$
3. $\displaystyle\sum_{j=0}^{5}2^j$
4. $\displaystyle\sum_{k=0}^{2}(3k-5)x^k$
5. $\displaystyle\sum_{i=1}^{4}\frac{1}{4}(i^2+1)$
6. $\displaystyle\sum_{n=1}^{100}(-1)^n$
7. $\displaystyle\sum_{n=1}^{5}\frac{(n+1)!}{n!}$
8. $\displaystyle\sum_{j=1}^{3}\frac{5!}{j!\,(5-j)!}$

In Exercises 9 - 16, rewrite the sum using summation notation.

9. $8+11+14+17+20$
10. $1-2+3-4+5-6+7-8$
11. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$
12. $1+2+4+\cdots+2^{29}$
13. $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}$
14. $-\ln(3)+\ln(4)-\ln(5)+\cdots+\ln(20)$
15. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}$
16. $\frac{1}{2}(x-5)+\frac{1}{4}(x-5)^2+\frac{1}{6}(x-5)^3+\frac{1}{8}(x-5)^4$

In Exercises 17 - 28, use the formulas in Equation 9.2 to find the sum.

17. $\displaystyle\sum_{n=1}^{10}5n+3$
18. $\displaystyle\sum_{n=1}^{20}2n-1$
19. $\displaystyle\sum_{k=0}^{15}3-k$
20. $\displaystyle\sum_{n=1}^{10}\left(\frac{1}{2}\right)^n$
21. $\displaystyle\sum_{n=1}^{5}\left(\frac{3}{2}\right)^n$
22. $\displaystyle\sum_{k=0}^{5}2\left(\frac{1}{4}\right)^k$
23. $1+4+7+\ldots+295$
24. $4+2+0-2-\ldots-146$
25. $1+3+9+\ldots+2187$
26. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{256}$
27. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+-\cdots+\frac{3}{256}$
28. $\displaystyle\sum_{n=1}^{10}-2n+\left(\frac{5}{3}\right)^n$

In Exercises 29 - 32, use Theorem 9.2 to express each repeating decimal as a fraction of integers.

29. $0.\overline{7}$
30. $0.\overline{13}$
31. $10.\overline{159}$
32. $-5.8\overline{67}$

In Exercises 33 - 38, use Equation 9.3 to compute the future value of the annuity with the given terms. In all cases, assume the payment is made monthly, the interest rate given is the annual rate, and interest is compounded monthly.

33. payments are \$300, interest rate is 2.5%, term is 17 years.

34. payments are \$50, interest rate is 1.0%, term is 30 years.

35. payments are \$100, interest rate is 2.0%, term is 20 years

36. payments are \$100, interest rate is 2.0%, term is 25 years

37. payments are \$100, interest rate is 2.0%, term is 30 years

38. payments are \$100, interest rate is 2.0%, term is 35 years

39. Suppose an ordinary annuity offers an annual interest rate of 2%, compounded monthly, for 30 years. What should the monthly payment be to have \$100,000 at the end of the term?

40. Prove the properties listed in Theorem 9.1.

41. Show that the formula for the future value of an annuity due is

$$A = P(1+i)\left[\frac{(1+i)^{nt}-1}{i}\right]$$

42. Discuss with your classmates what goes wrong when trying to find the following sums.[8]

(a) $\displaystyle\sum_{k=1}^{\infty} 2^{k-1}$ (b) $\displaystyle\sum_{k=1}^{\infty} (1.0001)^{k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}$

[8]When in doubt, write them out!

9.2.2 Answers

1. 213
2. $\frac{341}{280}$
3. 63
4. $-5 - 2x + x^2$
5. $\frac{17}{2}$
6. 0
7. 20
8. 25
9. $\displaystyle\sum_{k=1}^{5}(3k+5)$
10. $\displaystyle\sum_{k=1}^{8}(-1)^{k-1}k$
11. $\displaystyle\sum_{k=1}^{4}(-1)^{k-1}\frac{x}{2k-1}$
12. $\displaystyle\sum_{k=1}^{30}2^{k-1}$
13. $\displaystyle\sum_{k=1}^{5}\frac{k+1}{k}$
14. $\displaystyle\sum_{k=3}^{20}(-1)^{k}\ln(k)$
15. $\displaystyle\sum_{k=1}^{6}\frac{(-1)^{k-1}}{k^2}$
16. $\displaystyle\sum_{k=1}^{4}\frac{1}{2k}(x-5)^k$
17. 305
18. 400
19. -72
20. $\dfrac{1023}{1024}$
21. $\dfrac{633}{32}$
22. $\dfrac{1365}{512}$
23. 14652
24. -5396
25. 3280
26. $\dfrac{255}{256}$
27. $\dfrac{513}{256}$
28. $\dfrac{17771050}{59049}$
29. $\dfrac{7}{9}$
30. $\dfrac{13}{99}$
31. $\dfrac{3383}{333}$
32. $-\dfrac{5809}{990}$
33. \$76,163.67
34. \$20,981.40
35. \$29,479.69
36. \$38,882.12
37. 49,272.55
38. 60,754.80
39. For \$100,000, the monthly payment is $\approx$ \$202.95.

9.3 Mathematical Induction

The Chinese philosopher Confucius is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here we introduce a method of proof, Mathematical Induction, which allows us to *prove* many of the formulas we have merely *motivated* in Sections 9.1 and 9.2 by starting with just a single step. A good example is the formula for arithmetic sequences we touted in Equation 9.1. Arithmetic sequences are defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and we go from one term to the next by successively adding d. In symbols,

$$a, a+d, a+2d, a+3d, a+4d+\ldots$$

The pattern *suggested* here is that to reach the nth term, we start with a and add d to it exactly $n-1$ times, which lead us to our formula $a_n = a + (n-1)d$ for $n \geq 1$. But how do we *prove* this to be the case? We have the following.

The Principle of Mathematical Induction (PMI): Suppose $P(n)$ is a sentence involving the natural number n.

IF

1. $P(1)$ is true **and**
2. whenever $P(k)$ is true, it follows that $P(k+1)$ is also true

THEN the sentence $P(n)$ is true for all natural numbers n.

The Principle of Mathematical Induction, or PMI for short, is exactly that - a principle.[1] It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove that a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the '**base step**') and then show that if it is true for a generic natural number k, it must be true for the next natural number, $k+1$ (the '**inductive step**'). The notation $P(n)$ acts just like function notation. For example, if $P(n)$ is the sentence (formula) '$n^2 + 1 = 3$', then $P(1)$ would be '$1^2 + 1 = 3$', which is false. The construction $P(k+1)$ would be '$(k+1)^2 + 1 = 3$'. As usual, this new concept is best illustrated with an example. Returning to our quest to prove the formula for an arithmetic sequence, we first identify $P(n)$ as the formula $a_n = a + (n-1)d$. To prove this formula is valid for all natural numbers n, we need to do two things. First, we need to establish that $P(1)$ is true. In other words, is it true that $a_1 = a + (1-1)d$? The answer is yes, since this simplifies to $a_1 = a$, which is part of the definition of the arithmetic sequence. The second thing we need to show is that whenever $P(k)$ is true, it follows that $P(k+1)$ is true. In other words, we *assume* $P(k)$ is true (this is called the '**induction hypothesis**') and *deduce* that $P(k+1)$ is also true. Assuming $P(k)$ to be true seems to invite disaster - after all, isn't this essentially what we're trying to prove in the first place? To help explain this step a little better, we show how this works for specific values of n. We've already established $P(1)$ is true, and we now want to show that $P(2)$

[1] Another word for this you may have seen is 'axiom.'

is true. Thus we need to show that $a_2 = a+(2-1)d$. Since $P(1)$ is true, we have $a_1 = a$, and by the definition of an arithmetic sequence, $a_2 = a_1 + d = a + d = a+(2-1)d$. So $P(2)$ is true. We now use the fact that $P(2)$ is true to show that $P(3)$ is true. Using the fact that $a_2 = a+(2-1)d$, we show $a_3 = a+(3-1)d$. Since $a_3 = a_2 + d$, we get $a_3 = (a+(2-1)d)+d = a+2d = a+(3-1)d$, so we have shown $P(3)$ is true. Similarly, we can use the fact that $P(3)$ is true to show that $P(4)$ is true, and so forth. In general, if $P(k)$ is true (i.e., $a_k = a+(k-1)d$) we set out to show that $P(k+1)$ is true (i.e., $a_{k+1} = a+((k+1)-1)d$). Assuming $a_k = a+(k-1)d$, we have by the definition of an arithmetic sequence that $a_{k+1} = a_k + d$ so we get $a_{k+1} = (a+(k-1)d)+d = a+kd = a+((k+1)-1)d$. Hence, $P(k+1)$ is true.

In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on. Thus as long as $P(k)$ is true for some natural number k, $P(n)$ is true for all of the natural numbers n which follow k. Coupling this with the fact $P(1)$ is true, we have established $P(k)$ is true for all natural numbers which follow $n = 1$, in other words, all natural numbers n. One might liken Mathematical Induction to a repetitive process like climbing stairs.[2] If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase.[3] We get some more practice with induction in the following example.

Example 9.3.1. Prove the following assertions using the Principle of Mathematical Induction.

1. The sum formula for arithmetic sequences: $\displaystyle\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$.
2. For a complex number z, $(\overline{z})^n = \overline{z^n}$ for $n \geq 1$.
3. $3^n > 100n$ for $n > 5$.
4. Let A be an $n \times n$ matrix and let A' be the matrix obtained by replacing a row R of A with cR for some real number c. Use the definition of determinant to show $\det(A') = c\det(A)$.

Solution.

1. We set $P(n)$ to be the equation we are asked to prove. For $n = 1$, we compare both sides of the equation given in $P(n)$

$$\begin{aligned} \sum_{j=1}^{1}(a+(j-1)d) &\stackrel{?}{=} \frac{1}{2}(2a+(1-1)d) \\ a+(1-1)d &\stackrel{?}{=} \frac{1}{2}(2a) \\ a &= a \checkmark \end{aligned}$$

[2] Falling dominoes is the most widely used metaphor in the mainstream College Algebra books.
[3] This is how Carl climbed the stairs in the Cologne Cathedral. Well, that, and encouragement from Kai.

This shows the base case $P(1)$ is true. Next we assume $P(k)$ is true, that is, we assume

$$\sum_{j=1}^{k}(a+(j-1)d) = \frac{k}{2}(2a+(k-1)d)$$

and attempt to use this to show $P(k+1)$ is true. Namely, we must show

$$\sum_{j=1}^{k+1}(a+(j-1)d) = \frac{k+1}{2}(2a+(k+1-1)d)$$

To see how we can use $P(k)$ in this case to prove $P(k+1)$, we note that the sum in $P(k+1)$ is the sum of the first $k+1$ terms of the sequence $a_k = a+(k-1)d$ for $k \geq 1$ while the sum in $P(k)$ is the sum of the first k terms. We compare both side of the equation in $P(k+1)$.

$$\begin{aligned}
\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{summing the first } k+1 \text{ terms}} &\stackrel{?}{=} \frac{k+1}{2}(2a+(k+1-1)d) \\
\underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{summing the first } k \text{ terms}} + \underbrace{(a+(k+1-1)d)}_{\text{adding the } (k+1)\text{st term}} &\stackrel{?}{=} \frac{k+1}{2}(2a+kd) \\
\underbrace{\frac{k}{2}(2a+(k-1)d)}_{\text{Using } P(k)} + (a+kd) &\stackrel{?}{=} \frac{(k+1)(2a+kd)}{2} \\
\frac{k(2a+(k-1)d)+2(a+kd)}{2} &\stackrel{?}{=} \frac{2ka+k^2d+2a+kd}{2} \\
\frac{2ka+2a+k^2d+kd}{2} &= \frac{2ka+2a+k^2d+kd}{2} \checkmark
\end{aligned}$$

Since all of our steps on both sides of the string of equations are reversible, we conclude that the two sides of the equation are equivalent and hence, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have that $P(n)$ is true for all natural numbers n.

2. We let $P(n)$ be the formula $(\overline{z})^n = \overline{z^n}$. The base case $P(1)$ is $(\overline{z})^1 = \overline{z^1}$, which reduces to $\overline{z} = \overline{z}$ which is true. We now assume $P(k)$ is true, that is, we assume $(\overline{z})^k = \overline{z^k}$ and attempt to show that $P(k+1)$ is true. Since $(\overline{z})^{k+1} = (\overline{z})^k\,\overline{z}$, we can use the induction hypothesis and

write $(\overline{z})^k = \overline{z^k}$. Hence, $(\overline{z})^{k+1} = (\overline{z})^k\ \overline{z} = \overline{z^k}\,\overline{z}$. We now use the product rule for conjugates[4] to write $\overline{z^k}\,\overline{z} = \overline{z^k z} = \overline{z^{k+1}}$. This establishes $(\overline{z})^{k+1} = \overline{z^{k+1}}$, so that $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $(\overline{z})^n = \overline{z^n}$ for all $n \geq 1$.

3. The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, this means our base step occurs at $n = 6$. We can still use the PMI in this case, but our conclusion will be that the formula is valid for all $n \geq 6$. We let $P(n)$ be the inequality $3^n > 100n$, and check that $P(6)$ is true. Comparing $3^6 = 729$ and $100(6) = 600$, we see $3^6 > 100(6)$ as required. Next, we assume that $P(k)$ is true, that is we assume $3^k > 100k$. We need to show that $P(k+1)$ is true, that is, we need to show $3^{k+1} > 100(k+1)$. Since $3^{k+1} = 3 \cdot 3^k$, the induction hypothesis gives $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k$. We are done if we can show $300k > 100(k+1)$ for $k \geq 6$. Solving $300k > 100(k+1)$ we get $k > \frac{1}{2}$. Since $k \geq 6$, we know this is true. Putting all of this together, we have $3^{k+1} = 3 \cdot 3^k > 3(100k) = 300k > 100(k+1)$, and hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$.

4. To prove this determinant property, we use induction on n, where we take $P(n)$ to be that the property we wish to prove is true for all $n \times n$ matrices. For the base case, we note that if A is a 1×1 matrix, then $A = [a]$ so $A' = [ca]$. By definition, $\det(A) = a$ and $\det(A') = ca$ so we have $\det(A') = c\det(A)$ as required. Now suppose that the property we wish to prove is true for all $k \times k$ matrices. Let A be a $(k+1) \times (k+1)$ matrix. We have two cases, depending on whether or not the row R being replaced is the first row of A.

   CASE 1: The row R being replaced is the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p}$$

   where the $1p$ cofactor of A' is $C'_{1p} = (-1)^{(1+p)} \det\left(A'_{1p}\right)$ and A'_{1p} is the $k \times k$ matrix obtained by deleting the 1st row and pth column of A'.[5] Since the first row of A' is c times the first row of A, we have $a'_{1p} = c\,a_{1p}$. In addition, since the remaining rows of A' are identical to those of A, $A'_{1p} = A_{1p}$. (To obtain these matrices, the first row of A' is removed.) Hence $\det\left(A'_{1p}\right) = \det\left(A_{1p}\right)$, so that $C'_{1p} = C_{1p}$. As a result, we get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} c\,a_{1p} C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c\det(A),$$

   as required. Hence, $P(k+1)$ is true in this case, which means the result is true in this case for all natural numbers $n \geq 1$. (You'll note that we did not use the induction hypothesis at all in this case. It is possible to restructure the proof so that induction is only used where

[4] See Exercise 54 in Section 3.4.
[5] See Section 8.5 for a review of this notation.

it is needed. While mathematically more elegant, it is less intuitive, and we stand by our approach because of its pedagogical value.)

Case 2: The row R being replaced is the not the first row of A. By definition,

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p},$$

where in this case, $a'_{1p} = a_{1p}$, since the first rows of A and A' are the same. The matrices A'_{1p} and A_{1p}, on the other hand, are different but in a very predictable way − the row in A'_{1p} which corresponds to the row cR in A' is exactly c times the row in A_{1p} which corresponds to the row R in A. In other words, A'_{1p} and A_{1p} are $k \times k$ matrices which satisfy the induction hypothesis. Hence, we know $\det\left(A'_{1p}\right) = c \det\left(A_{1p}\right)$ and $C'_{1p} = c\, C_{1p}$. We get

$$\det(A') = \sum_{p=1}^{n} a'_{1p} C'_{1p} = \sum_{p=1}^{n} a_{1p} c\, C_{1p} = c \sum_{p=1}^{n} a_{1p} C_{1p} = c \det(A),$$

which establishes $P(k+1)$ to be true. Hence by induction, we have shown that the result holds in this case for $n \geq 1$ and we are done. □

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our main use of this result comes in Section 9.4 to prove the celebrated Binomial Theorem. The ardent Mathematics student will no doubt see the PMI in many courses yet to come. Sometimes it is explicitly stated and sometimes it remains hidden in the background. If ever you see a property stated as being true 'for all natural numbers n', it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

9.3.1 EXERCISES

In Exercises 1 - 7, prove each assertion using the Principle of Mathematical Induction.

1. $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

3. $2^n > 500n$ for $n > 12$

4. $3^n \geq n^3$ for $n \geq 4$

5. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$

6. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.

7. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for $n \geq 1$.

8. Prove Equations 9.1 and 9.2 for the case of geometric sequences. That is:

   (a) For the sequence $a_1 = a$, $a_{n+1} = ra_n$, $n \geq 1$, prove $a_n = ar^{n-1}$, $n \geq 1$.

   (b) $\displaystyle\sum_{j=1}^{n} ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$, $\displaystyle\sum_{j=1}^{n} ar^{n-1} = na$, if $r = 1$.

9. Prove that the determinant of a lower triangular matrix is the product of the entries on the main diagonal. (See Exercise 8.3.1 in Section 8.3.) Use this result to then show $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

10. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

9.3.2 Selected Answers

1. Let $P(n)$ be the sentence $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. For the base case, $n=1$, we get

$$\begin{aligned} \sum_{j=1}^{1} j^2 &\stackrel{?}{=} \frac{(1)(1+1)(2(1)+1)}{6} \\ 1^2 &= 1 \checkmark \end{aligned}$$

   We now assume $P(k)$ is true and use it to show $P(k+1)$ is true. We have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 &\stackrel{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ \sum_{j=1}^{k} j^2 + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{Using } P(k)} + (k+1)^2 &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)+6(k+1)^2}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k(2k+1)+6(k+1))}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)\left(2k^2+7k+6\right)}{6} &\stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k+2)(2k+3)}{6} &= \frac{(k+1)(k+2)(2k+3)}{6} \checkmark \end{aligned}$$

   By induction, $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all natural numbers $n \geq 1$.

4. Let $P(n)$ be the sentence $3^n > n^3$. Our base case is $n=4$ and we check $3^4 = 81$ and $4^3 = 64$ so that $3^4 > 4^3$ as required. We now assume $P(k)$ is true, that is $3^k > k^3$, and try to show $P(k+1)$ is true. We note that $3^{k+1} = 3 \cdot 3^k > 3k^3$ and so we are done if we can show $3k^3 > (k+1)^3$ for $k \geq 4$. We can solve the inequality $3x^3 > (x+1)^3$ using the techniques of Section 5.3, and doing so gives us $x > \frac{1}{\sqrt[3]{3}-1} \approx 2.26$. Hence, for $k \geq 4$, $3^{k+1} = 3 \cdot 3^k > 3k^3 > (k+1)^3$ so that $3^{k+1} > (k+1)^3$. By induction, $3^n > n^3$ is true for all natural numbers $n \geq 4$.

6. Let $P(n)$ be the sentence $\log(x^n) = n\log(x)$. For the duration of this argument, we assume $x > 0$. The base case $P(1)$ amounts checking that $\log\left(x^1\right) = 1\log(x)$ which is clearly true. Next we assume $P(k)$ is true, that is $\log\left(x^k\right) = k\log(x)$ and try to show $P(k+1)$ is true. Using the Product Rule for Logarithms along with the induction hypothesis, we get

$$\log\left(x^{k+1}\right) = \log\left(x^k \cdot x\right) = \log\left(x^k\right) + \log(x) = k\log(x) + \log(x) = (k+1)\log(x)$$

Hence, $\log\left(x^{k+1}\right) = (k+1)\log(x)$. By induction $\log(x^n) = n\log(x)$ is true for all $x > 0$ and all natural numbers $n \geq 1$.

9. Let A be an $n \times n$ lower triangular matrix. We proceed to prove the $\det(A)$ is the product of the entries along the main diagonal by inducting on n. For $n = 1$, $A = [a]$ and $\det(A) = a$, so the result is (trivially) true. Next suppose the result is true for $k \times k$ lower triangular matrices. Let A be a $(k+1) \times (k+1)$ lower triangular matrix. Expanding $\det(A)$ along the first row, we have

$$\det(A) = \sum_{p=1}^{n} a_{1p}C_{1p}$$

Since $a_{1p} = 0$ for $2 \leq p \leq k+1$, this simplifies $\det(A) = a_{11}C_{11}$. By definition, we know that $C_{11} = (-1)^{1+1}\det\left(A_{11}\right) = \det\left(A_{11}\right)$ where A_{11} is $k \times k$ matrix obtained by deleting the first row and first column of A. Since A is lower triangular, so is A_{11} and, as such, the induction hypothesis applies to A_{11}. In other words, $\det\left(A_{11}\right)$ is the product of the entries along A_{11}'s main diagonal. Now, the entries on the main diagonal of A_{11} are the entries a_{22}, a_{33}, $\ldots$, $a_{(k+1)(k+1)}$ from the main diagonal of A. Hence,

$$\det(A) = a_{11}\det\left(A_{11}\right) = a_{11}\left(a_{22}a_{33}\cdots a_{(k+1)(k+1)}\right) = a_{11}a_{22}a_{33}\cdots a_{(k+1)(k+1)}$$

We have $\det(A)$ is the product of the entries along its main diagonal. This shows $P(k+1)$ is true, and, hence, by induction, the result holds for all $n \times n$ upper triangular matrices. The $n \times n$ identity matrix I_n is a lower triangular matrix whose main diagonal consists of all 1's. Hence, $\det\left(I_n\right) = 1$, as required.

9.4 The Binomial Theorem

In this section, we aim to prove the celebrated **Binomial Theorem**. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In Elementary and Intermediate Algebra, you should have seen specific instances of the formula, namely

$$\begin{array}{rcl}(a+b)^1 & = & a+b \\ (a+b)^2 & = & a^2+2ab+b^2 \\ (a+b)^3 & = & a^3+3a^2b+3ab^2+b^3\end{array}$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4 = (a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4 = (a+b)\left(a^3+3a^2b+3ab^2+b^3\right) = a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as $(a+b)^{k+1} = (a+b)(a+b)^k$. Clearly this means Mathematical Induction plays a major role in the proof of the Binomial Theorem.[1] Before we can state the theorem we need to revisit the sequence of factorials which were introduced in Example 9.1.1 number 6 in Section 9.1.

Definition 9.4. Factorials: For a whole number n, ***n* factorial**, denoted $n!$, is the term f_n of the sequence $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$.

Recall this means $0! = 1$ and $n! = n(n-1)!$ for $n \geq 1$. Using the recursive definition, we get: $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Informally, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ with $0! = 1$ as our 'base case.' Our first example familiarizes us with some of the basic computations involving factorials.

Example 9.4.1.

1. Simplify the following expressions.

   (a) $\dfrac{3!\,2!}{0!}$ (b) $\dfrac{7!}{5!}$ (c) $\dfrac{1000!}{998!\,2!}$ (d) $\dfrac{(k+2)!}{(k-1)!}$, $k \geq 1$

2. Prove $n! > 3^n$ for all $n \geq 7$.

Solution.

1. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.

   (a) We have been programmed to react with alarm to the presence of a 0 in the denominator, but in this case $0! = 1$, so the fraction is defined after all. As for the numerator, $3! = 3 \cdot 2 \cdot 1 = 6$ and $2! = 2 \cdot 1 = 2$, so we have $\frac{3!\,2!}{0!} = \frac{(6)(2)}{1} = 12$.

[1] It's pretty much the reason Section 9.3 is in the book.

(b) We have $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ while $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Dividing, we get $\frac{7!}{5!} = \frac{5040}{120} = 42$. While this is correct, we note that we could have saved ourselves some of time had we proceeded as follows

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42$$

In fact, should we want to fully exploit the recursive nature of the factorial, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42$$

(c) Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

(d) This problem continues the theme which we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator. To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

2. We proceed by induction and let $P(n)$ be the inequality $n! > 3^n$. The base case here is $n = 7$ and we see that $7! = 5040$ is larger than $3^7 = 2187$, so $P(7)$ is true. Next, we assume that $P(k)$ is true, that is, we assume $k! > 3^k$ and attempt to show $P(k+1)$ follows. Using the properties of the factorial, we have $(k+1)! = (k+1)k!$ and since $k! > 3^k$, we have $(k+1)! > (k+1)3^k$. Since $k \geq 7$, $k+1 \geq 8$, so $(k+1)3^k \geq 8 \cdot 3^k > 3 \cdot 3^k = 3^{k+1}$. Putting all of this together, we have $(k+1)! = (k+1)k! > (k+1)3^k > 3^{k+1}$ which shows $P(k+1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$. □

Of all of the mathematical animals we have discussed in the text, factorials grow most quickly. In problem 2 of Example 9.4.1, we proved that $n!$ overtakes 3^n at $n = 7$. 'Overtakes' may be too polite a word, since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show. It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$.[2]

Applications of factorials in the wild often involve counting arrangements. For example, if you have fifty songs on your mp3 player and wish arrange these songs in a playlist in which the order of the

[2]This fact is far more important than you could ever possibly imagine.

songs matters, it turns out that there are 50! different possible playlists. If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!}$ such playlists. If, on the other hand, you just want to select ten song files out of the fifty to put on a flash memory card so that now the order no longer matters, there are $\frac{50!}{40!10!}$ ways to achieve this.[3] While some of these ideas are explored in the Exercises, the authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics and Statistics. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as our next definition indicates.

Definition 9.5. Binomial Coefficients: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\boldsymbol{\binom{n}{j}}$ (read, n choose j) is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

The name 'binomial coefficient' will be justified shortly. For now, we can physically interpret $\binom{n}{j}$as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you won two free tickets to a special screening of the latest Hollywood blockbuster and have five good friends each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying Definition 9.5, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends have to stay home.) The reader is encouraged to verify this by actually taking the time to list all of the possibilities.

We now state anf prove a theorem which is crucial to the proof of the Binomial Theorem.

Theorem 9.3. For natural numbers n and j with $n \geq j$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$$

The proof of Theorem 9.3 is purely computational and uses the definition of binomial coefficients, the recursive property of factorials and common denominators.

[3]For reference,

$$\begin{aligned} 50! &= 30414093201713378043612608166064768844377641568960512000000000000, \\ \frac{50!}{40!} &= 37276043023296000, \quad \text{and} \\ \frac{50!}{40!10!} &= 10272278170 \end{aligned}$$

$$\begin{aligned}
\binom{n}{j-1} + \binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!} + \frac{n!}{j(j-1)!(n-j)!} \\
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!} + \frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\
&= \frac{n!\,j}{j!(n-j+1)!} + \frac{n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,j + n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\
&= \frac{(n+1)n!}{j!(n+1-j))!} \\
&= \frac{(n+1)!}{j!((n+1)-j))!} \\
&= \binom{n+1}{j} \checkmark
\end{aligned}$$

We are now in position to state and prove the Binomial Theorem where we see that binomial coefficients are just that - coefficients in the binomial expansion.

Theorem 9.4. Binomial Theorem: For nonzero real numbers a and b,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

for all natural numbers n.

To get a feel of what this theorem is saying and how it really isn't as hard to remember as it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$\begin{aligned}(a+b)^4 &= \sum_{j=0}^{4}\binom{4}{j}a^{4-j}b^j\\ &= \binom{4}{0}a^{4-0}b^0+\binom{4}{1}a^{4-1}b^1+\binom{4}{2}a^{4-2}b^2+\binom{4}{3}a^{4-3}b^3+\binom{4}{4}a^{4-4}b^4\\ &= \binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4\end{aligned}$$

We forgo the simplification of the coefficients in order to note the pattern in the expansion. First note that in each term, the total of the exponents is 4 which matched the exponent of the binomial $(a+b)^4$. The exponent on a begins at 4 and decreases by one as we move from one term to the next while the exponent on b starts at 0 and increases by one each time. Also note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$ whereas the lower number changes from term to term and matches the exponent of b in that term. This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)(a+b)(a+b)(a+b)$, our answer is the sum of all possible products with exactly four factors - some a, some b. If we wish to count, for instance, the number of ways we obtain 1 factor of b out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be a, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1}a^3b$ is in the expansion. The other terms which appear cover the remaining cases. While this discussion gives an indication as to *why* the theorem is true, a formal proof requires Mathematical Induction.[4]

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem and we note that $P(1)$ is true since

$$\begin{aligned}(a+b)^1 &\stackrel{?}{=} \sum_{j=0}^{1}\binom{1}{j}a^{1-j}b^j\\ a+b &\stackrel{?}{=} \binom{1}{0}a^{1-0}b^0+\binom{1}{1}a^{1-1}b^1\\ a+b &= a+b\,\checkmark\end{aligned}$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in Theorem 9.4 and attempt to show that $P(k+1)$ is true.

[4]and a fair amount of tenacity and attention to detail.

$$
\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j + b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}
\end{aligned}
$$

Our goal is to combine as many of the terms as possible within the two summations. As the counter j in the first summation runs from 0 through k, we get terms involving a^{k+1}, $a^k b$, $a^{k-1}b^2$, ..., ab^k. In the second summation, we get terms involving $a^k b$, $a^{k-1}b^2$, ..., ab^k, b^{k+1}. In other words, apart from the first term in the first summation and the last term in the second summation, we have terms common to both summations. Our next move is to 'kick out' the terms which we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j$$

and

$$\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

so that

$$(a+b)^{k+1} = a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} + b^{k+1}$$

We now wish to write

$$\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}$$

as a single summation. The wrinkle is that the first summation starts with $j = 1$, while the second starts with $j = 0$. Even though the sums produce terms with the same powers of a and b, they do so for different values of j. To resolve this, we need to shift the index on the second summation so that the index j starts at $j = 1$ instead of $j = 0$ and we make use of Theorem 9.1 in the process.

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=0+1}^{k-1+1}\binom{k}{j-1}a^{k-(j-1)}b^{(j-1)+1}\\ &= \sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\end{aligned}$$

We can now combine our two sums using Theorem 9.1 and simplify using Theorem 9.3

$$\begin{aligned}\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\left[\binom{k}{j}+\binom{k}{j-1}\right]a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}\end{aligned}$$

Using this and the fact that $\binom{k+1}{0}=1$ and $\binom{k+1}{k+1}=1$, we get

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+b^{k+1}\\ &= \binom{k+1}{0}a^{k+1}b^{0}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+\binom{k+1}{k+1}a^{0}b^{k+1}\\ &= \sum_{j=0}^{k+1}\binom{k+1}{j}a^{(k+1)-j}b^{j}\end{aligned}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers n.

Example 9.4.2. Use the Binomial Theorem to find the following.

1. $(x-2)^4$
2. 2.1^3
3. The term containing x^3 in the expansion $(2x+y)^5$

Solution.

1. Since $(x-2)^4=(x+(-2))^4$, we identify $a=x$, $b=-2$ and $n=4$ and obtain

$$\begin{aligned}(x-2)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j}(-2)^j \\ &= \binom{4}{0} x^{4-0}(-2)^0 + \binom{4}{1} x^{4-1}(-2)^1 + \binom{4}{2} x^{4-2}(-2)^2 + \binom{4}{3} x^{4-3}(-2)^3 + \binom{4}{4} x^{4-4}(-2)^4 \\ &= x^4 - 8x^3 + 24x^2 - 32x + 16\end{aligned}$$

2. At first this problem seem misplaced, but we can write $2.1^3 = (2+0.1)^3$. Identifying $a = 2$, $b = 0.1 = \frac{1}{10}$ and $n = 3$, we get

$$\begin{aligned}\left(2 + \frac{1}{10}\right)^3 &= \sum_{j=0}^{3} \binom{3}{j} 2^{3-j} \left(\frac{1}{10}\right)^j \\ &= \binom{3}{0} 2^{3-0} \left(\frac{1}{10}\right)^0 + \binom{3}{1} 2^{3-1} \left(\frac{1}{10}\right)^1 + \binom{3}{2} 2^{3-2} \left(\frac{1}{10}\right)^2 + \binom{3}{3} 2^{3-3} \left(\frac{1}{10}\right)^3 \\ &= 8 + \frac{12}{10} + \frac{6}{100} + \frac{1}{1000} \\ &= 8 + 1.2 + 0.06 + 0.001 \\ &= 9.261\end{aligned}$$

3. Identifying $a = 2x$, $b = y$ and $n = 5$, the Binomial Theorem gives

$$(2x+y)^5 = \sum_{j=0}^{5} \binom{5}{j} (2x)^{5-j} y^j$$

Since we are concerned with only the term containing x^3, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on x is 3, the exponent on y must be 2. Plucking out the term $j = 2$, we get

$$\binom{5}{2} (2x)^{5-2} y^2 = 10(2x)^3 y^2 = 80x^3y^2$$

□

We close this section with Pascal's Triangle, named in honor of the mathematician Blaise Pascal. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & & & \searrow\swarrow & & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & & & \searrow\swarrow & & \searrow\swarrow & & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all whole numbers n, we get that each row of Pascal's Triangle begins and ends with 1. To generate the numbers in the middle of the rows (from the third row onwards), we take advantage of the additive relationship expressed in Theorem 9.3. For instance, $\binom{1}{0} + \binom{1}{1} = \binom{2}{1}$, $\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$ and so forth. This relationship is indicated by the arrows in the array above. With these two facts in hand, we can quickly generate Pascal's Triangle. We start with the first two rows, 1 and 1 1. From that point on, each successive row begins and ends with 1 and the middle numbers are generated using Theorem 9.3. Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
 & & \searrow\swarrow & & \\
\boxed{1} & & \underline{1+1} & & \boxed{1}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & 2 & & 1
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \\
\boxed{1} & & \underline{1+2} & & \underline{2+1} & & \boxed{1}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
 & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\
\boxed{1} & & \underline{1+3} & & \underline{3+3} & & \underline{3+1} & & \boxed{1}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x - y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers which form the fifth row of Pascal's Triangle. Since we know that the exponent of $3x$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $-y$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned} (3x - y)^4 &= (1)(3x)^4 + (4)(3x)^3(-y) + (6)(3x)^2(-y)^2 + 4(3x)(-y)^3 + 1(-y)^4 \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4 \end{aligned}$$

We would like to stress that Pascal's Triangle is a very quick method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is definitely the way to go.

9.4.1 Exercises

In Exercises 1 - 9, simplify the given expression.

1. $(3!)^2$
2. $\dfrac{10!}{7!}$
3. $\dfrac{7!}{2^3 3!}$
4. $\dfrac{9!}{4!3!2!}$
5. $\dfrac{(n+1)!}{n!}$, $n \geq 0$.
6. $\dfrac{(k-1)!}{(k+2)!}$, $k \geq 1$.
7. $\dbinom{8}{3}$
8. $\dbinom{117}{0}$
9. $\dbinom{n}{n-2}$, $n \geq 2$

In Exercises 10 - 13, use Pascal's Triangle to expand the given binomial.

10. $(x+2)^5$
11. $(2x-1)^4$
12. $\left(\frac{1}{3}x + y^2\right)^3$
13. $\left(x - x^{-1}\right)^4$

In Exercises 14 - 17, use Pascal's Triangle to simplify the given power of a complex number.

14. $(1+2i)^4$
15. $\left(-1 + i\sqrt{3}\right)^3$
16. $\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)^3$
17. $\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)^4$

In Exercises 18 - 22, use the Binomial Theorem to find the indicated term.

18. The term containing x^3 in the expansion $(2x-y)^5$
19. The term containing x^{117} in the expansion $(x+2)^{118}$
20. The term containing $x^{\frac{7}{2}}$ in the expansion $(\sqrt{x} - 3)^8$
21. The term containing x^{-7} in the expansion $\left(2x - x^{-3}\right)^5$
22. The constant term in the expansion $\left(x + x^{-1}\right)^8$
23. Use the Prinicple of Mathematical Induction to prove $n! > 2^n$ for $n \geq 4$.
24. Prove $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = 2^n$ for all natural numbers n. (HINT: Use the Binomial Theorem!)
25. With the help of your classmates, research Patterns and Properties of Pascal's Triangle.
26. You've just won three tickets to see the new film, '8.$\overline{9}$.' Five of your friends, Albert, Beth, Chuck, Dan, and Eugene, are interested in seeing it with you. With the help of your classmates, list all the possible ways to distribute your two extra tickets among your five friends. Now suppose you've come down with the flu. List all the different ways you can distribute the three tickets among these five friends. How does this compare with the first list you made? What does this have to do with the fact that $\binom{5}{2} = \binom{5}{3}$?

9.4.2 Answers

1. 36
2. 720
3. 105
4. 1260
5. $n+1$
6. $\frac{1}{k(k+1)(k+2)}$
7. 56
8. 1
9. $\frac{n(n-1)}{2}$
10. $(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
11. $(2x-1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$
12. $\left(\frac{1}{3}x + y^2\right)^3 = \frac{1}{27}x^3 + \frac{1}{3}x^2y^2 + xy^4 + y^6$
13. $\left(x - x^{-1}\right)^4 = x^4 - 4x^2 + 6 - 4x^{-2} + x^{-4}$
14. $-7 - 24i$
15. 8
16. i
17. -1
18. $80x^3y^2$
19. $236x^{117}$
20. $-24x^{\frac{7}{2}}$
21. $-40x^{-7}$
22. 70

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